Quick summary on "Klassinen dynamiikka"

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I. LAGRANGIAN DYNAMICS

- **Generalized coordinates:** You don't have to work with Cartesian coordinates. Think about the problem and choose the coordinates that are most suitable. Try to have as few as possible. By convention generalized coordinates are usually denoted as q_σ where σ runs from 1 to degrees of freedom ($3N-$ number of constraints).
- **Constraints:** These are conditions which constrain the motion. We mostly work with holonomic constraints of type

$$f_j(x_1,\dots x_n) = c_j \;\; j = 1,2,\dots k \tag{1}$$

  where $n = 3N$ is the number of degrees of freedom in the absence of constraints.
- **Virtual displacement:** Infinitesimal and instantaneous displacement of coordinates consistent with constraints.

$$\delta x_i = \sum_{\sigma}^{n-k} \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma \tag{2}$$

- **D'Alembert's principle:**: Forces of constraint (reaction forces), do no work under virtual displacement.
- **Lagrange's equation:** This follows from D'Alembert's principle

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0}, \tag{3}$$

  where for conservative forces with potential $V(q_1 \dots q_{n-k})$, $L = T - V$ with T the kinetic energy. There is one equation for each q_σ.

  If q_σ does not appear in the Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0 \tag{4}$$

  and $\partial L/\partial \dot{q}_\sigma$ is a constant of motion. Such generalized coordinate is known as **cyclic** and the quantity $\partial L/\partial \dot{q}_\sigma$ is known as the **canonical (or generalized) momentum**.

II. CALCULUS OF VARIATIONS

Basic idea is to find a function, let us say $y(x)$, so that the integral (functional)

$$I = \int_{x_1}^{x_2} \phi(y(x), y'(x), x)dx \tag{5}$$

is an extremum. Examples, minimize the distance between points, or in higher dimensions surface area... Fermat's principle: light travels along a path that minimizes the travel time. When we look for the extremum we find an **Euler-Lagrange equation** for the function $y(x)$

$$\boxed{\frac{d}{dt}\frac{\partial \phi}{\partial y'} - \frac{\partial \phi}{\partial y} = 0}. \tag{6}$$

Be sure to understand how this was derived! The idea generalizes easily to functionals involving many different functions $y_i(x)$. We get an Euler-Lagrange equation for each degree of freedom.

A. Variational problems with constraints

Often you might have a variational problem with functional

$$I = \int_{x_1}^{x_2} \phi(y(x), y'(x), x)dx \tag{7}$$

together with a constraint (fixed mass, volume, length...whatever)

$$C = \int_{x_1}^{x_2} G(y(x), y'(x), x)dx. \tag{8}$$

It can be shown that this problem is solved by finding a solution $y(x)$ and a constant λ such that the augmented functional

$$I_2 = \int_{x_1}^{x_2} \left[\phi(y(x), y'(x), x) + \lambda G(y(x), y'(x), x)\right] dx \tag{9}$$

takes an extremal. λ is known as **Lagrange multiplier** and can be determined for example by inserting the solution $y(x)$ (a function of λ) to a constraint and solving for λ.

III. HAMILTON'S PRINCIPLE AND FORCES OF CONSTRAINT

In classical mechanics the relevant functional is **the action** which is the time integral of Lagrangian

$$S = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t] \tag{10}$$

Hamilton's principle amounts to demanding that the $q(t)$ must be such that the action is an extremum. This is also known as the principle of stationary action. Then the Euler-Lagrange equation for the extremum is nothing but our earlier Lagrange's equation

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0} \tag{11}$$

Be sure to understand the derivation of this.

If you need forces of constraint, you can use **Lagrange multipliers** λ_j. One for each constraint.

Variations of constraints

$$f_j(q_1, \ldots q_n) = c_j \;\; j = 1, 2, \ldots k \tag{12}$$

are zero and therefore can be incorporated into action. You then have set of Lagrange equations with the constraints

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma}}. \tag{13}$$

If we have n, generalized coordinates this amounts to n equations. We get k more equations from the constraints allowing the determination of all unknown quantities. Right hand side of the above equation can be identified as the generalized reaction force for q_σ.

IV. GENERALIZED MOMENTUM/CANONICAL MOMENTUM

We define generalized or canonical momentum via

$$\boxed{p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}}. \tag{14}$$

This is not always the same as the linear momentum. Using this definition Lagrange equation becomes

$$\frac{d}{dt}p_\sigma = \frac{\partial L}{\partial q_\sigma}. \tag{15}$$

If q_σ doesn't appear in the lagrangian it is called a **cyclic coordinate**. In that case the corresponding generalized momentum is a constant of motion.

V. CONSERVATION LAWS

There is a conserved quantity for each continuous symmetry (Noether's theorem). A symmetry transformation means something that leaves the system i.e. lagrangian unchanged. More precisely we can transform the generalized coordinates as

$$q_\sigma \to q_\sigma + \epsilon K_\sigma(q_1 \ldots q_n). \tag{16}$$

If the set of K's is such that lagrangian is unchanged to 1st order in ϵ, then the system is invariant and has this symmetry. In this case $dL(\epsilon)/d\epsilon = 0$ from which a conserved quantity follows (derivation + Lagrange equation)

$$P(q_1 \ldots q_n; \dot{q}_1 \ldots \dot{q}_n) = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} K_\sigma \tag{17}$$

VI. HAMILTONIAN DYNAMICS

- **Hamiltonian**

$$\boxed{H = \sum_\sigma p_\sigma \dot{q}_\sigma - L}, \tag{18}$$

  where $p_\sigma = \partial L / \partial \dot{q}_\sigma$ is the canonical momentum. Unlike Lagrangian this is a constant of motion if Lagrangian has no explicit time-dependence. Be sure to understand why. Derivation is not hard, but requires you understand what canonical momenta and Lagrange's equations mean.

  If the potential and constraints are time-independent, Hamiltonian is the systems total energy and a constant of motion. This is common, but not guaranteed.

- **Hamilton's equations:** pair of first order equations of q_σ and p_σ (view these as independent variables in the Hamiltonian...rather than q_σ and $\dot{q}_\sigma$).

$$\boxed{\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}} \tag{19}$$

$$\boxed{\dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}} \tag{20}$$

- **Phase space:** This is a space spanned by $(q_1 \ldots q_n, p_1 \ldots p_n)$. Each point represents a possible state and dynamics can be described as flow in this space.

- **Poisson bracket** of functions $F(q_1 \ldots q_n, p_1 \ldots p_n, t)$ and $G(q_1 \ldots q_n, p_1 \ldots p_n, t)$ is defined by

$$\boxed{[F, G]_{pq} = \{F, G\} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right)} \tag{21}$$

  Using this (and Hamilton's equations) the time-evolution of F can be expressed by

$$\frac{dF}{dt} = -\{H, F\} + \frac{\partial F}{\partial t} \tag{22}$$

  This has strong similarity to Heisenberg's equations of motion for an operator $\hat{F}$ in quantum mechanics

$$\frac{d\hat{F}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{F}] + \frac{\partial \hat{F}}{\partial t}, \tag{23}$$

  were $[,]$ is the quantum mechanical commutator. Description of classical mechanics in terms of Poisson brackets was an important spring board to quantum mechanics, where Poisson brackets end up being replaced by commutators (and some constant in front + bunch of other details).

- **Canonical transformation**: It might be helpful to redefine your variables $q_\sigma \to Q_\sigma$, $p_\sigma \to P_\sigma$. If you do that, in general,the dynamical equations in the new coordinates do not look like Hamilton's equations. However, if we demand that the dynamical equations must keep the same form i.e.

$$\boxed{\dot{Q}_\sigma = \frac{\partial K}{\partial P_\sigma}} \tag{24}$$

$$\boxed{\dot{P}_\sigma = -\frac{\partial K}{\partial Q_\sigma}}, \tag{25}$$

where K is the Hamiltonian in the new coordinates, transformation is **canonical**

- Consequence of the previous point: Transformation $q_\sigma \to Q_\sigma$, $p_\sigma \to P_\sigma$ is **canonical** if it preserves Poisson brackets $[p_\alpha, q_\beta]_{pq}$.

- Another consequence of the previous point: Under canonical transformation, area of the phase space is preserved. This ends up connecting with the Poisson bracket since Poisson bracket corresponds to the Jacobian when we transform the area integral into new coordinates!

- **Generating functions**: When we applied Hamilton's principle and looked for the extremum of

$$S = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t] = \int_{t_1}^{t_2} dt \sum_\sigma p_\sigma \dot{q}_\sigma - H(q, p), \tag{26}$$

we found Euler-Lagrange equations which also corresponded to the Hamilton's equations. However, we can add a total derivative of time to the integrand and the solution to the optimization problem doesn't change.

So we can make a point transformation $Q_\sigma = Q_\sigma(q, p, t)$ and $P_\sigma = P_\sigma(q, p, t)$ and add a derivative $L' = L + dF(q, t)/dt$ without changing Euler-Lagrange equations. Hamilton's principle in the beginning and after transformations are

$$\delta \int_{t_1}^{t_2} dt (\sum_\sigma p_\sigma \dot{q}_\sigma - H(q, p, t)) = 0 \tag{27}$$

and

$$\delta \int_{t_1}^{t_2} dt (\sum_\sigma P_\sigma \dot{Q}_\sigma - K(Q, P, t)) = 0 \tag{28}$$

For restractied canonical transformation there is no explicit time dependence in the transformation $(q, p) \to (Q, P)$ and we have

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(q, p, t) = P_\sigma \dot{Q}_\sigma - K(Q, P, t) + \dot{F} \tag{29}$$

If we choose $F = F_1(q, Q, t)$, this becomes

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(q, p, t) = P_\sigma \dot{Q}_\sigma - K(Q, P, t) + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}. \tag{30}$$

From which it follows

$$p_\sigma = \frac{\partial F_1}{\partial q_\sigma} \quad \text{and} \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} \quad \text{and} \quad K = H + \frac{\partial F_1}{\partial t} \tag{31}$$

giving us the point transformation for the coordinates and the connection to new Hamiltonian. Function $F_1(q, Q, t)$ is the **generating function for the canonical transformation**.

We have other generating functions as well depending on which coordinates it involves. Changing the variables is done via Legendre transformation. For example, $F = F_2(q, P, t) - \sum_\sigma Q_\sigma P_\sigma$, which gives

$$p_\sigma = \frac{\partial F_2}{\partial q_\sigma} \quad \text{and} \quad Q_\sigma = \frac{\partial F_2}{\partial P_\sigma} \quad \text{and} \quad K = H + \frac{\partial F_2}{\partial t} \tag{32}$$

$F = F_3(p, Q, t) + \sum_\sigma q_\sigma p_\sigma$, which gives

$$q_\sigma = -\frac{\partial F_3}{\partial p_\sigma} \quad \text{and} \quad P_\sigma = -\frac{\partial F_3}{\partial Q_\sigma} \quad \text{and} \quad K = H + \frac{\partial F_3}{\partial t} \tag{33}$$

And finally $F = F_4(p, P, t) + \sum_\sigma q_\sigma p_\sigma - Q_\sigma P_\sigma$, which gives

$$q_\sigma = -\frac{\partial F_4}{\partial p_\sigma} \quad \text{and} \quad Q_\sigma = \frac{\partial F_4}{\partial P_\sigma} \quad \text{and} \quad K = H + \frac{\partial F_4}{\partial t} \tag{34}$$

- Charged particle (charge e) in an electromagnetic field: Lagrangian given by (SI units)

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m\dot{\mathbf{r}}^2 - e\phi(\mathbf{r}, t) + e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t), \tag{35}$$

$\phi(\mathbf{r}, t)$ is the scalar potential and $\mathbf{A}(\mathbf{r}, t)$ the vector potential. From these magnetic field can be computed as $\mathbf{B} = \nabla \times \mathbf{A}$ and electric field as $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$.

Note that here generalized momentum becomes

$$\mathbf{p} = \frac{\partial L}{\partial \dot{r}} = m\dot{\mathbf{r}} + e\mathbf{A}(\mathbf{r}, t) \tag{36}$$

while the Hamiltonian is

$$H = \frac{1}{2m} \left[\mathbf{p} - e\mathbf{A}(\mathbf{r}, t)\right]^2 + e\phi(\mathbf{r}, t) \tag{37}$$

If potentials have no z dependence (for example), z-component of generalized momentum is conserved. However, the conserved quantity contains contribution from the vector potential.

VII. SPECIAL RELATIVITY

Starting points:

1. Laws of nature should be the same in all inertial frames
2. Everyone agrees on the speed of light. It is same in all inertial frames

Maxwell's laws are not invariant under Galilei transformation. We can derive wave equation that predict electromagnetic waves traveling at te speed of light from the Maxwell's equations, but non-invariance with respect to Galilei transformation suggests that speed of light should appear different for different observers traveling at constant velocity with respect to each other.

Michelson-Morley experiment observed no such thing. Speed if light appeared to be constant no matter which way relative to earths motion it was measured. There seems to be something wrong in how we transform between inertial coordinates!

Maxwell's equations are invariant if we **ditch Galilei transformations and use Lorentz transformations** instead. If the relative motion between frames is along x at velocity v, the Lorentz transformation is given by

$$\begin{aligned} t' &= \gamma(t - \frac{vx}{c^2}) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z, \end{aligned} \tag{38}$$

where c is the speed of light and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor. For small velocities compared to speed of light, it approaches Galilei transformation, but is generally different. Also, note that it changes time as well. In other words time becomes relative to since clocks can run at different rates in different inertial frames.

There are many important consequences from this:

- Events can be causally connected only if light had time tio move between them.

- Time dilatation: if the clock moving with the object measures a time interval of Δt, in the lab frame where the object appears to move, a lonfer time, $\Delta t' = \gamma \Delta t$ has passed. Proper time τ (itseisaika in finnish) is special in that it is measured by a clock attached to an object. It is related to the lab time via $\tau = t/\gamma$

- Length contraction: in the frame where an object is moving, it appears to be shorter, than in its rest frame. $L' = L/\gamma$

- Energy and mass are connected:

$$E = \sqrt{(pc)^2 + (m_0 c^2)^2} \tag{39}$$

  This implies that at rest when $p = 0$, $E = m_0 c^2$. When velocity is small compared to speed of light, we can Taylor expand the dispersion and get

$$E \approx m_0 c^2 + \frac{p^2}{2m_0} \tag{40}$$

  which is the usual kinetic energy, but now corrected with the (typically huge) rest energy.

- Newton's laws must be adjusted since there is no absolute time. However, lot of these changes are included when we adjust the mass of the particle so that it become velocity dependent i.e.

$$m = m_0 / \sqrt{1 - v^2/c^2}. \tag{41}$$

  Now mass increases indefinitely as we approach the speed of light. This prevents massive particles from ever reaching the speed of light. Relativistically acceptable Lagrangian for a free particle with mass m_0 turns out to be

$$L = -m_0 c^2 \sqrt{1 - (v/c)^2}, \tag{42}$$

  where v is the particle speed.

- It becomes natural to describe world with 4-dimensions. One for time and 3 for space. It can be done in Newtonian world as well, but since time is absolute there, it is kind of pointless there.

- In relativity things are often expressed in terms of 4-vectors. These are four component objects $\mathbf{X} = (X_0, X_1, X_2, X_3)$ that transform like time and position under Lorentz transformation. For example 4-position would be

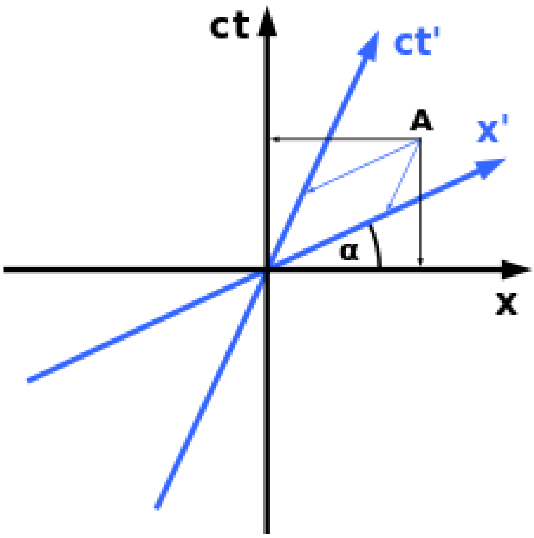
$$\mathbf{X} = (ct, x, y, z). \tag{43}$$

  The norm of the 4-vector is given by $||X||^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2$. It can be easily shown that this is the same in all inertial frames.

  Other 4-vectors include 4-velocity (derivative of 4-position with respect to proper time), 4-momentum (multiply 4-velocoty my rest mass), 4-current, and 4-potential.

- Events can be naturally marked in a **space-time diagram**. The Lorentz transformation into a moving frame implies that space and time axis are not orthogonal. This implies that, for example, the time ordering of the events may change when we look at them in a new frame. Speed of light is constant and its movement therefore always amounts to a diagonal line at 45 degrees.

  Light lines create light cones for the future and the past. Only events inside the light cones (inside meaning closer to the time axis) can be reached with signals traveling at less than the spped of light. Therefore, only those events can be causally connected.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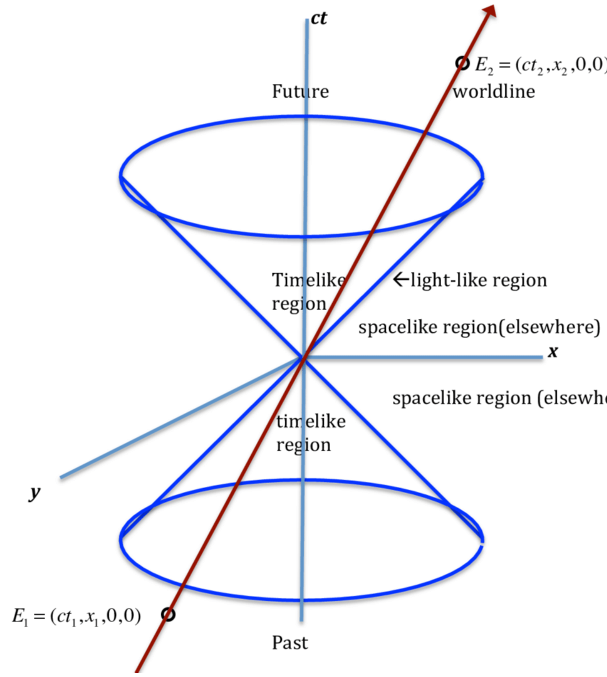
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worldline
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Timelike region
←light-like region
spacelike region(elsewhere)
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timelike region
y
$E_1 = (ct_1, x_1, 0, 0)$
Past