Mixed Integer Linear Programming (MILP)

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Contents

• Mixed integer linear programming (MILP)
  – Formulation
  – Graphical representation
  – Sample energy models
  – Solving MILP models

• Dynamic LP/MILP modelling
  – Hybrid power, heat and cooling optimization for buildings
LP and MILP modelling

• Linear Programming and Mixed Integer Linear Programming are most commonly used approaches for practical problems because
  – the modelling techniques are very versatile and flexible
  – efficient and reliable solvers exist for these problems

• Arbitrary convex optimization problems can be approximated with LP models

• Many non-convex optimization problems can be approximated with MILP models
Non-convex optimization problems

• When the (minimized) objective function or some of the constraints are not convex, then the problem is non-convex

• A non-convex problem may have several local optima
  – In the general case it is not possible to know at beforehand which local optimum is the global optimum
    → necessary to explore them all
  – It can be difficult (and even impossible) to ensure that all local optima have been explored
  – In MILP problems this can be ensured!
Mixed integer linear programming (MILP) model

• A mixed integer linear programming problem is similar to an LP model, but some of the variables have integer domain:

\[
\min (\max) \, cx + dy
\]

s.t.

\[
Ax + By \leq b
\]
\[
x \geq 0
\]
\[
y_i \in \{0,1\} \text{ (or some other finite range of integers)}
\]

• If all variables are integers, the problem is a (pure) integer linear programming (ILP) problem
Properties of MILP models

• **Special case of non-convex problems**
  – Optimum is always at a corner point of an LP model that is obtained by fixing the integer variables to some feasible values

• **Let** \( \mathbf{y}^* = \) vector of 0/1 values
  – Then \( \mathbf{d} \mathbf{y}^* = \) constant and \( \mathbf{B} \mathbf{y} = \) constant vector
  – An LP model results

\[
\begin{align*}
\min (\max) \quad & \mathbf{c} \mathbf{x} + \text{constant} \\
\text{s.t.} \quad & A \mathbf{x} \leq \mathbf{b} - \mathbf{B} \mathbf{y} = \text{constant vector} \\
& \mathbf{x} \geq 0
\end{align*}
\]
Properties of MILP models

• Reliable (but not so efficient) solution algorithms exist
  – The Branch&Bound algorithm will enumerate explicitly or implicitly the different value combinations of integer variables
  – This reduces the MILP problem into multiple LP problems

• Finite non-convex problems can be approximated by MILP models with arbitrarily good accuracy
  – In principle a MILP model can always be solved
  – However, the resulting model may become large and very slow to solve
    • number of LP models to solve can be astronomical
How to define a MILP model?

1. Write down a verbal explanation of what is the goal or purpose of the model
   - E.g. to minimize costs or maximize profit from some specific operation or activity

2. Define the decision variables (and parameters)
   - Specify if they are real numbers or binary or general integers
   - Use as descriptive or generic names as you like: x1, x2, fuel, …
   - Give short description for them
   - Also specify the unit (MWh, GJ, €/kg, m3/s, …)

3. Define the objective function to minimize or maximize as a linear function of the variables

4. Define the constraints as linear inequality or equality constraints of the variables

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Example of energy MILP modelling

• Biofuel power plant that can be shut down
  – Plant operation follows a linear characteristic in a range
    \[ x_{el} = \frac{x_{bio}}{R} - P_{loss} \]
    \[ x_{el}^{\text{min}} \leq x_{el} \leq x_{el}^{\text{max}} \]
    
    \( x_{bio} \) = biofuel consumption
    \( R \) = consumption ratio for biofuel
    \( P_{loss} \) = constant loss
    \( x_{el}^{\text{min}}, x_{el}^{\text{max}} \) = minimum/maximum power production
  
  – But when the plant is shut down, \( x_{el} = x_{bio} = 0 \)
    • Because the characteristic is non-continuous, it is also non-convex
Biofuel power plant characteristic

Graph showing the relationship between fuel input $x_{bio}$ and power output $x_{el}$. The graph indicates a linear increase in power output as the fuel input increases.
Biofuel power plant MILP model

- A binary variable \( y \) is defined as a switch to determine if the plant is on \((y=1)\) or off \((y=0)\).
- Encoded model

\[
\begin{align*}
\text{Max } & \quad c\text{el}x\text{el} - c\text{bio}x\text{bio} \\
x\text{el} & = x\text{bio}/R - y*P\text{loss} \\
y*x\text{el}\text{min} & \leq x\text{el} \leq y*x\text{el}\text{max} \\
y & \in \{0,1\}
\end{align*}
\]

- The \( y \)-variable affects both the plant characteristic and bounds for power output.

\( c\text{el} = \) price for sold power
\( c\text{bio} = \) fuel price
Wrong ways to use binary variables

- Sometimes students people try to define:

  \[
  \text{Max } y^*(c_{el}x_{el} - c_{bio}x_{bio}) \quad \text{// not linear}
  \]
  \[
  x_{el} = y^*(x_{bio}/R - P_{loss}) \quad \text{// not linear}
  \]
  \[
  x_{el_{\text{min}}} \leq x_{el} \leq x_{el_{\text{max}}} \quad \text{// infeasible when plant is off}
  \]

  \[y \in \{0,1\}\]

  ♠ Objective is not linear, product of variables
  ♠ Constraint is not linear, product of variables
  ♠ Lower bound of \(x_{el}\) is infeasible when plant off

- Objective function and constraints in MILP model must be linear as in an LP model!
Encoding logical relations into MILP model

- All logical operators (\(\land, \lor, \neg, \ldots\)) can be encoded using binary variables and linear constraints:
  
  \[
  X = Y \land Z \rightarrow x \leq y; \ x \leq z; \ x \geq y+z-1
  \]
  
  \[
  X = Y \lor Z \rightarrow x \leq y+z; \ x \geq y; \ x \geq z
  \]
  
  \[
  X = \neg Y \rightarrow x = 1-y
  \]

- Arbitrarily complex logical expressions can be encoded recursively in parts:
  
  \[
  Y = (Y_1 \land \neg Y_2) \lor Y_3 \iff Y = Z \lor Y_3; \ Z = Y_1 \land \neg Y_2
  \]
  
  \[
  \rightarrow y \leq z+y_3; \ y \geq z; \ y \geq y_3;
  \]
  
  \[
  z \leq y_1; \ z \leq 1-y_2; \ z \geq y_1+(1-y_2)-1
  \]
MILP-encoding of general non-convex problems

• A non-convex optimization problem is of form

\[ \text{Min } f(x); \ s.t. \ x \in X \]

– where \( f() \) is a non-convex function,
– or \( X \) is a non-convex set,
– or both
MILP-encoding of non-convex constraints

• X is partitioned into convex subsets $X = \bigcup X_i$
• A binary variable $y_i$ is defined for each part
  – the part is enabled when $y_i=1$ and disabled when $y_i=0$
• Each subset is modelled by linear constraints
  \[ A_i x \leq b_i + M(1-y_i) \]
• Binary variables activate exactly one set of constraints at a time
  \[ \sum y_i = 1 \]
  \[ y_i \in \{0,1\} \]
MILP-encoding of non-convex objective function

- A non-convex objective function can be approximated by a piecewise linear non-convex function
- Example: \( \min f(x) = -x^2 \) in range \( x \in [-2,2] \)
  - Choose points \( (X_i, F_i) \) along function
  - define \( (x,f) \) as convex combination of linear segments using **continuous variables** \( x_i \) and
  - **binary variables** \( y_i \) to enable exactly one segment

\[
x = \sum_j x_i X_i \\
f = \sum x_i F_i \\
\sum x_i = 1 \\
x_i \leq y_i + y_{i+1} \\
\sum y_i = 1 \\
x_i \geq 0, y_i \in \{0,1\}
\]
Convex CHP model

- The power plant characteristic defines in the P-Q plane the feasible operating area area
  - \( p = \) power production, \( q = \) heat production, \( c = \) fuel cost
- We encode the model as a *convex combination* of extreme (corner) points

\[
\max cp - \sum_j c_j x_j
\]

s.t.
\[
\sum_j p_j x_j = p \quad /\!/ \text{variable power prod}
\]
\[
\sum_j q_j x_j = q \quad /\!/ \text{fixed heat demand}
\]
\[
\sum_j x_j = 1 \quad /\!/ \text{convex comb.}
\]
\[
x_j \geq 0
\]
Non-convex CHP model

• Necessary when either (or both)
  – The cost function is non-convex
  – P-Q the characteristic is non-convex
    • E.g. when it is necessary to optimize the shut-down of the plant

• Idea
  – Partition objective function into convex parts
  – Partition characteristic into convex parts
  – Use 0/1 variables to choose in which area to operate
Sample non-convex cogeneration model

Allocation of characteristic points to convex sub-areas

<table>
<thead>
<tr>
<th>Area</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
<th>P5</th>
<th>P6</th>
<th>P7</th>
<th>P8</th>
<th>P9</th>
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<tr>
<td>A1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
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<td>1</td>
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</tr>
<tr>
<td>A3</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>
Non-convex cogeneration model

• Characteristic is partitioned in three convex parts
• $A_j$ is set of areas to which $x_j$ belongs
• Define zero-one variables $y_1, y_2, y_3$, and allow exactly one of them to have value 1.
• $y$-variables select which corner points are allowed in the convex combination

<table>
<thead>
<tr>
<th>Area</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
<th>P5</th>
<th>P6</th>
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<tr>
<td>A3</td>
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</tr>
</tbody>
</table>

$$x_j \leq \sum_{a \in A_j} y_a, \quad j \in J_u, \quad u \in U^*,$$
$$\sum_{a \in A_u} y_a = 1, \quad u \in U^*,$$
$$y_a \in \{0, 1\}, \quad a \in A_u, \quad u \in U^*.$$
Solving MILP models

• In principle it is possible to solve MILP problems using brute force:
  – Choose a value combination of integer variables
  – Solve the resulting LP problem
  – The best feasible solution among all combinations gives the optimum

• The number of problems to solve is exponential with respect to number of variables
  – With N binary variables, there are $2^N$ combinations
  – $N=10 \rightarrow 1024$, $20 \rightarrow 10^6$, $30 \rightarrow 10^9$, …
Solving MILP models

• The Branch & Bound algorithm solves MILP models more efficiently by solving only a small fraction of all combinations
  – Still solution time may be exponential

• Standard software
  – CPLEX, GAMS, Lindo, Lingo, Excel Solver …

• Very efficient specialized algorithms exist for the extreme point formulation
  – Power Simplex, Extended Power Simplex, Tri-Commodity Simplex, …
One specialized algorithm for CHP

Available online at www.sciencedirect.com


Non-convex power plant modelling in energy optimisation

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Dynamic systems

• A dynamic system is one which develops in time
  – Opposite: static system

• Normally, a dynamic system is modelled by discretizing it into a sequence static models that are connected by dynamic constraints
Dynamic energy models

• Examples:
  – Yearly CHP planning model is represented by a sequence of 8760 hourly models
    • Dynamic constraints result from
      – energy storages
      – startup and shutdown costs and restrictions
  – Daily hydro power scheduling is represented by a sequence of 96 15min models
    • Dynamic constraints result from water level/amount in reservoirs and waterflows between reservoirs
Dynamic optimization

- Different ways to model and solve dynamic systems exist
  - **Multiperiod LP/MILP models**
    - Network flow model (LP)
  - General mathematical optimization models
  - Dynamic programming algorithm
  - Other network algorithms
A multiperiod LP/MILP model is an LP/MILP model with a special structure

- Time horizon is divided into a sequence of time periods, \( t = 1, \ldots, T \).
- The behaviour during each period \( t \) is modelled by a static LP/MILP model.
- The periods are connected by dynamic constraints linking together
  - subsequent period models pairwise, or
  - all period models for the entire horizon
Multiperiod LP modelling:
Subsequent constraints for energy storage

- $s_t = \text{storage level at end of period } t$
- $s_{\text{in}}^t, s_{\text{out}}^t = \text{charge & discharge during period } t$

- During each period, $s_t$ depends on previous level plus charge minus discharge
  
  $s_t = \eta^S s_{t-1} + \eta^{\text{in}} s_{\text{in}}^t - s_{\text{out}}^t / \eta^{\text{out}}, \ (t=1, \ldots, T)$

  $q_t - s_{\text{in}}^t + s_{\text{out}}^t = Q_t \ // \text{heat balance}$
  
  - $q_t$ is production of heat in period $t$
  - $Q_t$ is the demand for heat in period $t$
  - $\eta^S = \text{storage efficiency in time, } 1 \text{ if no loss}$
  - $\eta^{\text{in}} = \text{efficiency of charging storage}$
  - $\eta^{\text{out}} = \text{efficiency of discharging storage}$
  - $s_0 = \text{initial storage level (fixed, e.g. } 0 \text{ or } s_0 = s_T)$
Multiperiod MILP modelling:
Startup/shutdown costs and restrictions

• On/off status of power plant is represented by binary variables $y_t$ and startup/shutdown by $z^\text{up}_t$, $z^\text{down}_t$

\[
\begin{align*}
 z^\text{up}_t &\geq y_t - y_{t-1} \\
 z^\text{down}_t &\geq y_{t-1} - y_t; \quad (t = 1, \ldots, T)
\end{align*}
\]

- only $y_t$ need to be binary variables, $z^\text{up}$ & $z^\text{down}$ can be real
- $y_0$ is initial on/off status, which is fixed

• Startup/shutdown costs are included into objective

\[
\text{Min } \ldots + c^\text{up}_t z^\text{up}_t + c^\text{down}_t z^\text{down}_t;
\]

• Startup/shutdown restrictions are represented as logical constraints

\[
\begin{align*}
 z^\text{up}_t &\leq 1 - z^\text{down}_{t-1}; & &\text{// disable immediate startup} \\
 z^\text{up}_t &\leq 1 - z^\text{down}_{t-2}; & &\text{// and startup with 2-period delay etc.}
\end{align*}
\]

• A long-term model can be large and too complex to solve
Multiperiod MILP modelling example: Condensing power plant

- **Objective**: maximize profit during time horizon $t=1,…,T$
- **Plant**: can produce power between $x^{\text{min}}$ and $x^{\text{max}}$, or be shut down at a cost
- **Decision variables**
  - $x_t$: electric power production in period $t$
  - $y_t$: binary on/off variable for period $t$
  - $z_{t}^{\text{up}}, z_{t}^{\text{down}}$: startup & shutdown variable in period $t$
- **Parameters**
  - $c_{t}^{\text{el}}$: electric power sales price in period $t$
  - $c_{\text{fuel}}$: operating cost (fuel price)
  - $c_{\text{down}}$: shutdown cost
  - $c_{\text{on}}$: constant term for operating costs when running
  - $x^{\text{min}}, x^{\text{max}}$: min and max production when running

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Multiperiod MILP modelling example: Condensing power plant

- **Objective function**
  \[ \text{Max } \sum_{t=1,...,T} (c_{el}^t - c_{fuel}^t)x_t - c_{on}^t y_t - c_{down}^t z_{down}^t \]

- **Production constraints**
  \[ y_t^* x_{min} \leq x_t \leq y_t^* x_{max} \]

- **Startup & shutdown constraints**
  \[ z_{up}^t \geq y_t - y_{t-1} \] (not needed if no cost for startup)
  \[ z_{down}^t \geq y_{t-1} - y_t \]

- **Bounds for variables**
  \[ y_t \in \{0,1\} \]
  \[ z_{up}^t \geq 0 \]
  \[ z_{down}^t \geq 0 \]
Hybrid heat, power and cooling optimization for buildings

- Based on recent article
- Simultaneous optimization of configuration, sizing and operation of building energy system
- Three energy commodities: Electricity, heating, cooling
- Large multiperiod LP/MILP model with 8760-hour time horizon
Energy supply and storage optimization for mixed-type buildings

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Energy efficiency and renewable energy solutions in buildings is an important and actual research topic. Operating and fixed costs of sustainable energy solutions can be reduced by using optimization models. We developed a novel optimization model and applied it for a mixed-type building with commercial, office, and residential parts in Finland. The model determines the optimal configuration, dimensioning, and operation of different local energy production and storage technologies for power, heat, and cooling. The model is formulated as a large dynamic linear or mixed-integer linear programming model (LP/MILP) for a full year. The result shows that district heating, district cooling, energy storage, heat pumps, and photovoltaics as a hybrid solution for a building can both reduce the combined operating and fixed costs annually by 271,000€, and support meeting the nearly Zero Energy Building requirements with E-value limit of 107 kWh/m²/a. Photovoltaics can be profitable when consumed maximally at the building. While heat and cooling storages are cost-efficient for balancing demand and supply, power storages are still too expensive. District heating and heat pump heating worked synergetically together, but district cooling and heat pump cooling were mutually exclusive choices at nearly equal cost.

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Hybrid energy system for building

- District Heating
- Power Grid
- PV
- District Cooling

- Heat balance
- Power balance
- Cooling balance

- HP H&C
- HP Heat
- HP Cool

- Heat demand
- Heat Storage
- Power demand
- Power Storage
- Cooling demand
- Cooling Storage

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## Decision variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{u,t}$</td>
<td>MW</td>
<td>Hourly operating level of technology $u$</td>
</tr>
<tr>
<td>$s_{u,t}$</td>
<td>MWh</td>
<td>Hourly level of storage at end of period</td>
</tr>
<tr>
<td>$s_t^{\text{IN}}$, $s_t^{\text{OUT}}$</td>
<td>MW</td>
<td>Hourly storage charge/discharge</td>
</tr>
<tr>
<td>$x_u^{\text{MAX}}$</td>
<td>MW</td>
<td>Sizing (capacity) of each technology</td>
</tr>
<tr>
<td>$s^{\text{MAX}}$</td>
<td>MWh</td>
<td>Sizing (capacity) of each storage</td>
</tr>
<tr>
<td>$z_u$</td>
<td>1</td>
<td>Binary variables to determine if technology $u$ is included in configuration</td>
</tr>
</tbody>
</table>
Objective function

- Minimize total yearly fixed + operating costs
- Fixed costs = annuity for investments and yearly capacity costs for heat, cooling & power contracts
- Operating costs = production costs for local production + energy costs for purchased energy

\[
\min \sum_{u \in U} \sum_{t=1}^{T} c_{u,t} x_{u,t} + \sum_{u \in U} \left( c_{u}^{\text{MAX}} x_{u}^{\text{MAX}} + c_{u}^{\text{CONST}} z_{u} \right) + \sum_{u \in S} \left( c_{u}^{\text{MAX}} s_{u}^{\text{MAX}} + c_{u}^{\text{CONST}} z_{u} \right)
\]

1. U = set of energy supply technologies (units)
2. S = set of storages (P,H,C)
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Energy balance constraints

\[
\sum_{u \in U_{H}^{\text{CONTR}}} \chi_{u,t} + \sum_{u \in U_{H}^{\text{PROD}}} \chi_{u,t}^H - s_{H,t}^\text{IN} + s_{H,t}^\text{OUT} = d_{H,t}, \quad t = 1, \ldots, T
\]

\[
\sum_{u \in U_{C}^{\text{CONTR}}} \chi_{u,t} + \sum_{u \in U_{C}^{\text{PROD}}} \chi_{u,t}^C - s_{C,t}^\text{IN} + s_{C,t}^\text{OUT} = d_{C,t}, \quad t = 1, \ldots, T.
\]

\[
\sum_{u \in U_{P}^{+}} \chi_{u,t} - \sum_{u \in U_{P}^{-}} \chi_{u,t}^P - s_{P,t}^\text{IN} + s_{P,t}^\text{OUT} = d_{P,t}, \quad t = 1, \ldots, T.
\]

Superscript \( \text{CONTR} \) = external energy contract,
PROD = local production
Subscript \( P \) = Power, \( H \) = Heat, \( C \) = Cooling
Storage constraints

\[ s_{u,t} = \eta_u^S s_{u,t-1} + \eta_u^{IN} s_{u,t}^{IN} - s_{u,t}^{OUT} \frac{1}{\eta_u^{OUT}}, \]

\[ 0 \leq s_{u,t} \leq s_u^{MAX}, \]

\[ 0 \leq s_{u,t}^{IN} \leq s_u^{IN,MAX}, \]

\[ 0 \leq s_{u,t}^{OUT} \leq s_u^{OUT,MAX}, \quad t = 1, \ldots, T \]

\[ 0 \leq s_u^{MAX} \leq Fz_u^{ON}, \]

\[ z_u \in \{0, 1\}, \quad u \in S. \]
Results –
Fixed and operating costs in configurations

Annual costs, €/a

<table>
<thead>
<tr>
<th></th>
<th>OPER</th>
<th>FIXED</th>
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<tr>
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</tr>
<tr>
<td>NoStor</td>
<td>310000</td>
<td>260000</td>
</tr>
</tbody>
</table>
Results

• Cost optimization not only saves money, but also reduces CO2 emissions and makes the building meet nZEB (nearly Zero Energy Building) requirements
• Heat and cooling storages were very profitable, but power storages were not
• Photovoltaic power can be profitable when it is dimensioned so that nearly all production can be consumed within the building
Network flow model

- **A network consists of nodes and connecting arcs**
  - Some commodity (power, heat, …) can flow through the arcs
  - Normally arcs are **directed** allowing flow in only one direction
  - Two-way flow is represented as a pair of opposite arcs

- **Attributes** are associated to nodes and/or arcs
  - Supply/demand $\pm d_j$ of commodity at each node
  - Transfer **price** $c_{ij}$ through each arc
  - Possibly a maximum **capacity** $u_{ij}$ for each arc
The transshipment network flow model

- The aim is to determine flows $x_{ij}$ through each arc so that
  - All nodes in the network are balanced
    - For this to succeed, it is necessary that total supplies/demands at nodes $\sum_i d_i = 0$
  - Overall transportation costs are minimized
  - Transshipment = commodity can pass through other nodes before reaching its final goal
The transshipment network flow model

- LP-formulation

Min $\Sigma_i \Sigma_j c_{ij} x_{ij}$  \hspace{1cm} // minimize total costs

s.t.

$\Sigma_j x_{ij} - \Sigma_j x_{ji} = d_i$  \hspace{1cm} for each node i

$x_{ij} \geq 0$  \hspace{1cm} for each arc (i,j)

(optionally also capacity constraints $x_{ij} \leq u_{ij}$)

- Here

  - $x_{ij}$ is flow from node i to j
  - $c_{ij}$ is unit cost for flow from node i to j
  - $d_i$ is supply at node i, negative value = demand
  - $u_{ij}$ is maximum allowed flow from node i to j
The transshipment network flow model

• It is a special case of an LP model
• Can be solved using
  – generic Simplex algorithm for LP
  – much more efficient network simplex algorithm
• Applies to a wide variety of different problems
  – Piecewise linear convex production cost for commodity
  – Transfer of commodity between locations
  – Dynamic models with storages (without losses)
  – Applies also to combinations of the above