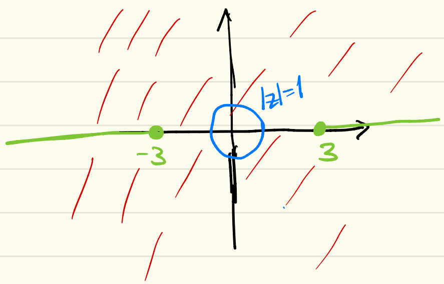
Exercise sheet 7

(1) By means of the local Cauchy Theorem (Cauchy's Theorem in a disk), assuming circles are positively oriented, calculate

a) $$\int_{|z|=1} \sqrt{9-z^2}\, dz$$

b) $$\int_{|z|=1} \frac{1}{z^2+2z}\, dz$$

<u>Solution</u>: a) The function $\sqrt{9-z^2}$ is analytic in the set where $\mathrm{Im}(9-z^2) \neq 0$ or where $\mathrm{Re}(9-z^2) > 0$ and $\mathrm{Im}(9-z^2) = 0$. That is in the red set.



Hence $$\int_{|z|=1} \sqrt{9-z^2}\, dz = 0$$ by Cauchy's Theorem

b) We use use partial fractions decomposition.

$$\frac{1}{z^2+2z} \overset{\text{Ansatz}}{=} \frac{A}{z} + \frac{B}{z+2} = \frac{(A+B)z+2A}{z(z+2)}$$

$$A = 1/2 \implies B = -1/2.$$

We know that $\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$ and

therefore

$$\int_{|z|=1} \frac{1}{z^2+2z}\,dz = \frac{1}{2}\int_{|z|=1} \frac{1}{z}\,dz - \frac{1}{2}\int_{|z|=1} \frac{1}{z+2}\,dz$$

$$= \pi i - \frac{1}{2}\int_{|z|=1} \frac{1}{z+2}\,dz \underset{\text{Cauchy}}{=} \pi i$$

(2) Let $\gamma(t) = 2\cos t + i\sin t$ for $0 \leq t \leq 2\pi$.

Evaluate:

a) $\int_\gamma \frac{1}{z}\,dz$.

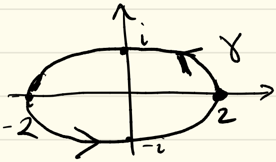
b) $\int_\gamma \frac{1}{z^2+2iz}\,dz$

<u>Hint</u>: Winding numbers help here

Solution: The path trajectory is an ellipse

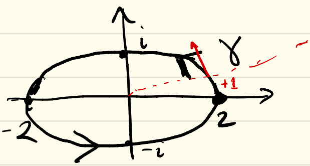
$$|\gamma| = \left\{ z = x+iy \; ; \; \frac{x^2}{4} + y^2 = 1 \right\}$$ since

$$\frac{(2\cos t)^2}{4} + \sin^2 t = 1.$$



a)

$$\int_\gamma \frac{1}{z}\, dz = 2\pi i\, n(\gamma, 0)$$



$n(\gamma,0) = 1$

$$\int_\gamma \frac{1}{z}\, dz = 2\pi i$$

b) Partial fractions

$$\frac{1}{z^2+2iz} \underset{\uparrow}{=} \frac{A}{z} + \frac{B}{z+2i} = \frac{(A+B)z+2iA}{z^2+2iz}$$

Ansatz

$$\Rightarrow A = -\frac{i}{2} \Rightarrow B = \frac{i}{2}$$

Note that $z=-2i$ is outside γ so

$$\int_\gamma \frac{B}{z+2i}\,dz = 0.$$

We get $\displaystyle\int_\gamma \frac{1}{z^2+2iz}\,dz = -\frac{i}{2}\int_\gamma \frac{1}{z}\,dz =$

$$= -\frac{i}{2}\,2\pi i = \pi$$

3 Calculate $\displaystyle\int_{|z+i|=3/2} \frac{1}{z^4+z^2}\,dz$ where $|z+i|=3/2$ is positively oriented

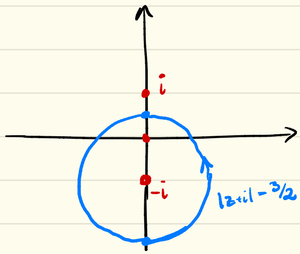
Solution

$$z^4+z^2 = z^2(z^2+1)$$

Therefore $\dfrac{1}{z^4+z^2}$ is analytic in

$$\mathbb{C}\setminus\{0, i, -i\}$$

We draw the trajectory $|z+i| = 3/2$



Partial fractions

Ansatz

$$\frac{1}{z^2(z+i)(z-i)} = \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z+i} + \frac{D}{z-i} =$$

$$= \frac{A(z+i)(z-i) + Bz(z+i)(z-i) + Cz^2(z-i) + Dz^2(z+i)}{z^2(z+i)(z-i)}$$

$$= \frac{(B+C+D)z^3 + (A - iC + iD)z^2 + Bz + A}{z^2(z+i)(z-i)}$$

$$\Longrightarrow \quad A = 1, \; B = 0, \; 1 + i(D - C) = 0, \; C + D = 0$$

$$\Longrightarrow \quad C = -D \;\; \& \;\; 1 + i2D = 0 \;\Rightarrow\; D = -\frac{1}{2i} = \frac{i}{2}$$

$$\Longrightarrow \quad \frac{1}{z^4 + z^2} = \frac{1}{z^2} + \frac{i}{2}\,\frac{1}{z-i} - \frac{i}{2}\,\frac{1}{z+i}$$

Since $\frac{d}{dz}\frac{1}{z} = \frac{1}{z^2}$ we get $\int_{|z+i| = 3/2} \frac{1}{z^2}\,dz = 0$

and since i is outside the curve we get

$$\int_{|z+i|=3/2} \frac{1}{z-i}\,dz = 0.$$

Hence $$\int_{|z+i|=3/2} \frac{1}{z^4+z^2}\,dz = -\frac{i}{2}\int_{|z+i|=3/2} \frac{1}{z+i}\,dz = -\frac{i}{2}\,2\pi i = \pi$$

(4) Let γ and β be closed, piecewise smooth paths in $\mathbb{C}$ with the same initial point. Show that $n(-\gamma, z) = -n(\gamma, z)$ for every $z \in \mathbb{C}\setminus|\gamma|$ and that

$$n(\gamma+\beta, z) = n(\gamma, z) + n(\beta, z)$$

for every $z \in \mathbb{C}\setminus(|\gamma|\cup|\beta|)$

<u>Solution</u>:

$$n(-\gamma, z) = \frac{1}{2\pi i}\int_{-\gamma} \frac{1}{\zeta - z}\,d\zeta =$$

$$= -\frac{1}{2\pi i}\int \frac{1}{\zeta - z}\,dz = -n(\gamma, z)$$

for every $z \in \mathbb{C}\setminus|\gamma|$

Also, $$n(\gamma+\beta, z) = \frac{1}{2\pi i}\int_{\gamma+\beta} \frac{1}{\zeta - z}\,d\zeta =$$

$$= \frac{1}{2\pi i}\int_{\gamma} \frac{1}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\beta} \frac{1}{\zeta - z}\,d\zeta = n(\gamma, z) + n(\beta, z)$$

for every $z \in \mathbb{C}\setminus(|\gamma|\cup|\beta|)$ $\varnothing$