# Exercise sheet 8

① Use the Cauchy integral formulas to evaluate the following contour integrals when the circles are positive oriented (that is, the winding numbers are 1 inside).

a) $\displaystyle\int_{|z|=1} \frac{\cos z}{z}\, dz$

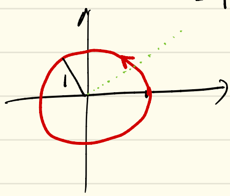
b) $\displaystyle\int_{|z|=2} \frac{e^{z+1}}{(z+1)^2}\, dz$

Solution: a) If $f(z) = \cos z$

Cauchy's Integral Formula gives

$$\int_{|z|=1} \frac{\cos z}{z}\, dz = \int \frac{f(z)}{z}\, dz = 2\pi i\, \underbrace{n(|z|=1, 0)}_{=1}\, f(0) =$$

$$= 2\pi i \cos 0 = 2\pi i$$



b) $f(z)=e^{z+1}$ is entire and therefore the Cauchy Integral Formula gives

$$n(|z|=2,-1)\, f'(-1) = \frac{1!}{2\pi i}\int_{|z|=2} \frac{f(z)}{(z-(-1))^2}\,dz .$$

Since $z=-1$ is inside $\{|z|=2\}$ and $f'(z)=e^{z+1}$ we get

$$\int_{|z|=2} \frac{e^z}{(z+1)^2}\,dz = 2\pi i\, e^{-1+1} = 2\pi i$$

(2) Let $f$ be an entire function with the property that $|f(z)| \le c|z|^{1/2}+d$ for all $z$, where $c$ and $d$ are positive constants. Prove that $f$ is a constant function.

Solution: Choose $z\in\mathbb{C}$.

Choose $r>0$ so that $z\in\Delta(0,r)$.

Then

$$f'(z) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta \quad \text{and}$$

$$|f'(z)| \le \frac{1}{2\pi}\int_{|\zeta|=r} \frac{|f(\zeta)|}{|\zeta-z|^2}\,|d\zeta| \le \frac{2\pi r\,(c r^{1/2}+d)}{r^2-|z|}$$

Since $3/2 < 2$ we get $\lim_{r\to\infty} \frac{2\pi r(cr^{1/2}+d)}{r^2-|z|} = 0$

and $f'(z)=0$

This holds for every $z \in \mathbb{C}$ so $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore $f(z)$ is a constant function.

(3) Let $u$ be a harmonic function in a disk $\Delta(0,r)$ (That is $u_{xx} + u_{yy} = 0$ in $\Delta(0,r)$). Use the Cauchy Integral Formula to show that

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{it})\, dt$$

when $0 < \rho < r$. You may assume that $u = \operatorname{Re}(f)$ in $\Delta(0,r)$ for some analytic function $f: \Delta(0,r) \to \mathbb{C}$.

Solution: We can find $v: \Delta(0,r) \to \mathbb{R}$ so that $f = u + iv : \Delta(0,r) \to \mathbb{C}$ is analytic.

By Cauchy's Integral Formula we have

$$f(0) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z}\, dz$$

Parametrize $\{|z| = \rho\}$ using $\gamma(t) = \rho e^{it}$, $0 \le t \le 2\pi$.

Then $\dot{\gamma}(t) = i\rho e^{it}\, dt$ and

$$f(0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\rho e^{it})}{\rho e^{it}}\, i\rho e^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it})\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{it}) + i v(\rho e^{it})\, dt$$

Take the real part of right and left hand side and get

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{it})\, dt$$

(4) If $f$ is a non-constant entire function, prove that the range $f(\mathbb{C})$ of $f$ must almost "fill up" the complex plane in the following sense:
for every point $w_0 \in \mathbb{C}$ and every $r > 0$ we have $f(\mathbb{C}) \cap \Delta(w_0, r) \neq \emptyset$. (We say that $f(\mathbb{C})$ is dense in $\mathbb{C}$). (Hint: Assume there is $w_0 \in \mathbb{C}$ and $r > 0$ such that $f(\mathbb{C}) \cap \Delta(w_0, r) = \emptyset$. Study $g(z) = (f(z) - w_0)^{-1}$ )

<u>Solution</u>: Assume there is $w_0 \in \mathbb{C}$ and $r > 0$ such that $f(\mathbb{C}) \cap \Delta(w_0, r) = \emptyset$.


Define $g(z) = \dfrac{1}{f(z) - w_0}$. This is analytic and defined for every $z \in \mathbb{C}$ (since $f(z) \neq w_0$) for every $z$. Since $f(\mathbb{C}) \cap \Delta(w_0, r) = \emptyset$ for every $z \in \mathbb{C}$ we see that $|f(z) - w_0| \geq r$ for all $z$. Therefore

$$|g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{r} \quad \text{for all } z \in \mathbb{C}$$

Liouville's Theorem implies that $g$ is constant. Therefore $f$ is also constant which is a contradiction.