Aalto university
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Exercise sheet 8
Complex Analysis, MS-C1300.
Hand in exercise 1 and 2 in separate files for grading. Deadline Wednesday 15.11 at 23:59. The exercises should be uploaded to the correct folder on MyCourses as pdf-files with name and student number in the file name. **Submission via MyCourses is the only accepted way.** Done during class Thursday 16.11 or Friday 17.11.

(1) Use the Cauchy integral formulas to evaluate the following contour integrals when the circles are positively oriented (that is, the winding numbers are 1 inside).

(a)
$$\int_{|z|=1} \frac{\cos z}{z}\, dz$$
(3p)

(b)
$$\int_{|z|=2} \frac{e^{z+1}}{(z+1)^2}\, dz$$
(3p)

(2) Let f be an entire function with the property that
$$|f(z)| \le c|z|^{1/2} + d$$
for all z, where λ, c, and d are positive constants. Prove that f is a constant function. (6p)

(3) Let $u\colon \Delta(0,r) \to \mathbb{R}$ be a harmonic function (that is $u_{xx} + u_{yy} = 0$ in $\Delta(0,r)$). Use Cauchys integral formula to show that
$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{it})\, dt$$
when $0 < \rho < r$. You may assume that $u = \mathrm{Re}(f)$ in $\Delta(0,r)$ for some analytic function $f\colon \Delta(0,r) \to \mathbb{C}$.

(4) If f is a non-constant entire function, prove that the range $f(\mathbb{C})$ of f must almost "fill up" the complex plane in the following sense: for every point $w_0 \in \mathbb{C}$ and every $r > 0$ we have $f(\mathbb{C}) \cap \Delta(w_0, r) \neq \emptyset$. (We say that $f(\mathbb{C})$ is dense in $\mathbb{C}$.) (*Hint*: Assume there is $w_0 \in \mathbb{C}$ and $r > 0$ such that $f(\mathbb{C}) \cap \Delta(w_0, r) = \emptyset$. Study $g(z) = 1/(f(z) - w_0)$.)