# ELEC-E8116 Model-based control systems
# /exercises with solutions 9

**Problem.** Consider the system

$$\frac{Y(s)}{U(s)} = \frac{s+0.5}{s^2+2s+4} \text{ and the criterion to be minimized } J = \int_0^\infty (3y^2 + 0.5u^2)\, dt \,.$$

Write a *Matlab m-file* to do the following:

Solve the optimal control law by using the *lqr*-function in Matlab. Calculate the *damping ratio* of the closed loop system. Simulate the system by letting the reference signal be zero (regulator problem) and letting the initial states be non-zero. Then consider the tracking problem. Use a static pre-compensator to set the static gain of the closed-loop system to the value 1. Then simulate the system for a step change in the reference signal.

**Solution.** Note that Matlab's *lqr* function uses the cost $J = \int_0^\infty \left( x^T Qx + u^T Ru \right) dt$

$$J = \int_0^\infty (3y^2 + 0.5u^2)dt = \int_0^\infty (3y^T y + 0.5u^2)dt = \int_0^\infty (3(Cx)^T Cx + 0.5u^2)dt$$

$$\int_0^\infty (x^T (3C^T C)x + 0.5u^2)dt$$

The Matlab m-code shows the solution and simulation.

The process model in state-space form is

```
A =

   -2   -2
    2    0

B =

    1
    0

C =

   1.0000    0.2500
```

D =

0

The process and optimal controller are

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -Lx$$

which leads to the closed loop representation

$$\dot{x} = (A - BL)x$$
$$y = Cx$$

The state feedback matrix, the Riccati equation solution and the closed loop poles are

L =

1.2197 0.0917

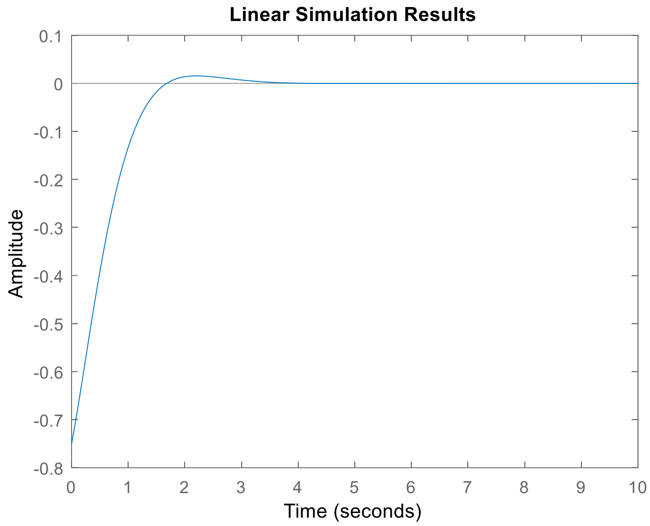
S =

0.6099 0.0458
0.0458 0.3086

E =

-1.6099 + 1.2616i
-1.6099 - 1.2616i

The damping ratio is 0.787. The simulation result using the initial condition [-1 1] is shown in the figure. Note that it shows the output signal *y*. If both states should be plotted, you should have defined the C-matrix as an identity matrix. The system would not change, but both states would be defined as outputs.




As for the servo problem, the controller is $u = -Lx + kr$ where $k$ is a constant and $r$ the reference. The closed loop is

$$\dot{x} = (A - BL)x + kBr$$
$$y = Cx$$

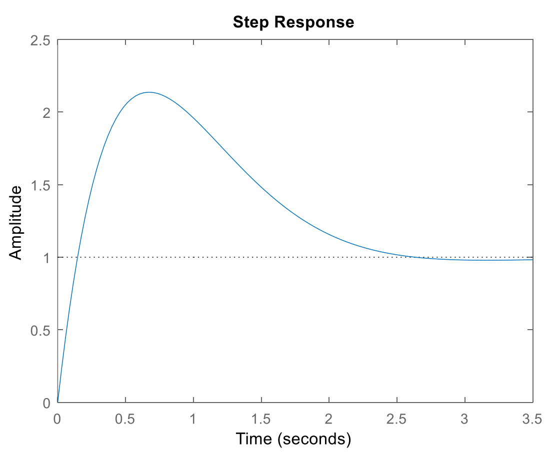
The corresponding transfer function from reference to output is

$$G_{cl}(s) = kC\left(sI - A + BL\right)^{-1} B$$

The static gain is $G_{cl}(0)$, and it must be set to 1 in order the output to follow the reference. Hence

$$k = \frac{1}{C(-A + BL)^{-1} B}$$

The simulation result for a step change in $r$ is shown below.



```matlab
% Model-based control systems
% Exercise 9, Problem
%
%Model
s=tf('s');
G=(s+0.5)/(s^2+2*s+4);
Gss=ss(G);
[A,B,C,D]=ssdata(Gss);
%Cost function
Q=3*C'*C; R=0.5;
%Optimal control u=-Lx, regulator problem
[L,S,E]=lqr(A,B,Q,R);
damp(A-B*L);
%Closed loop
Gclss=ss(A-B*L,zeros(2,1),C,0);
%Simulation
x0=[-1,1]';
T=0:0.01:10;
U=zeros(size(T));
lsim(Gclss,U,T,x0)
%Optimal control u=-Lx+kr, servo problem
k=1/(C*inv(-A+B*L)*B);
G2clss=ss(A-B*L,k*B,C,0);
figure
step(G2clss)
```