ELEC-E8116 Model-based control systems
Intermediate exam 1. 19. 10. 2023 / Solutions

Each problem 1-3 gives the maximum of 5 points.

1. Explain briefly the following concepts (max 1p. each)

- Singular value decomposition and singular values
- Bandwidth from the viewpoint of control engineering
- Minimal realization of a multivariable transfer function
- Internal stability
- Robust stability

Solution:

For any real or complex *mxn* matrix the exists the *singular value decomposition*

$$G = U\Sigma V^*$$

where the *mxm* and *nxn* matrices *U* and *V* are *unitary* ($UU^* = U^*U = I_m, VV^* = V^*V = I_n$) and the real *mxn* matrix Σ has the singular values in its main diagonal. If *G* is complex, then *U* and *V* are also complex, otherwise real. The singular values can also be calculated from

$$\sigma = \sqrt{\lambda(G^*G)} = \sqrt{\lambda(GG^*)}$$

where the non-zero eigenvalues are the same in both expressions.

The *bandwidth* generally defines the (angular) frequency band (or its upper value) where the output of a system follows sinusoidal input at that frequency. For higher frequencies the amplitude of the sinusoid at the output starts to mitigate. In control theory the bandwidth is usually linked to the performance of the closed loop denoting the frequency band where control is effective. In this respect there are different definitions for bandwidth (its maximum frequency): the gain crossover frequency of the loop transfer function, the frequency at which the sensitivity function crosses -3 dB from below, the frequency at which the complementary sensitivity function crosses -3 dB from above. (In MIMO systems the maximum singular value of the functions is used).

Minimal realization of a system transfer function is a state-space realization (representation) of it by using the minimum number of state variables. Then the representation is both controllable and observable. For SISO systems the minimal realization can easily be determined by first cancelling the same polynomials from numerator and denominator of the transfer function (pole-zero cancellations). Then the state-space representation using as many states as the degree of the denominator polynomial is minimal and therefore also controllable and observable.

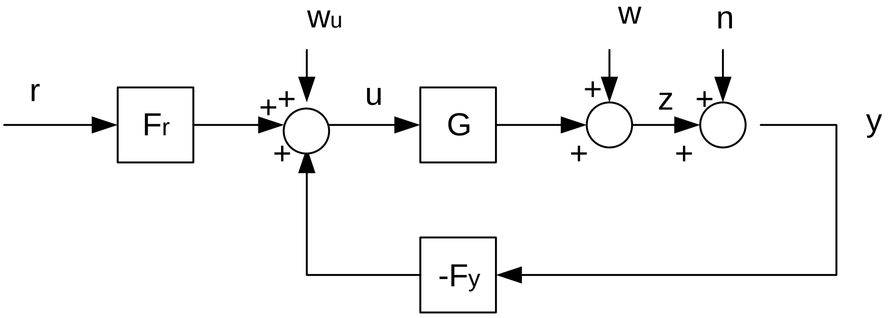
For MIMO systems the determination of minimum number of states is not trivial, since it is not easy to determine possible pole-zero cancellations. Help is given by *Karcanias theorems* (giving pole and zero polynomials that correspond the minimal realization).

The system is *internally stable* when no finite input signal (reference, disturbance) can lead to some output signal to grow without limit.

The system is *robustly stable*, when it remains stable for all values of the possible uncertainty in the system.

Note for problems 2 and 3: When writing equations with matrices remember that a square matrix can have inverse A^{-1} (if exists), but you cannot divide by a matrix (1/*A* is illegal operation for a matrix). Also note that for matrices $AB \neq BA$ except for some rare special cases.

2. Consider a **multivariable** control configuration in the below figure, where signal *y* is *m*-dimensional and signal *u* *n*-dimensional (*m* and *n* are positive integers).



a. What are the dimensions of signals r, w_u, w, n and matrices G, F_y, F_r? (1 p.)

b. Give the condition by which the 2 DOF (two degrees-of-freedom) control configuration in the figure changes into a 1 DOF configuration. Draw a figure. (2 p.)

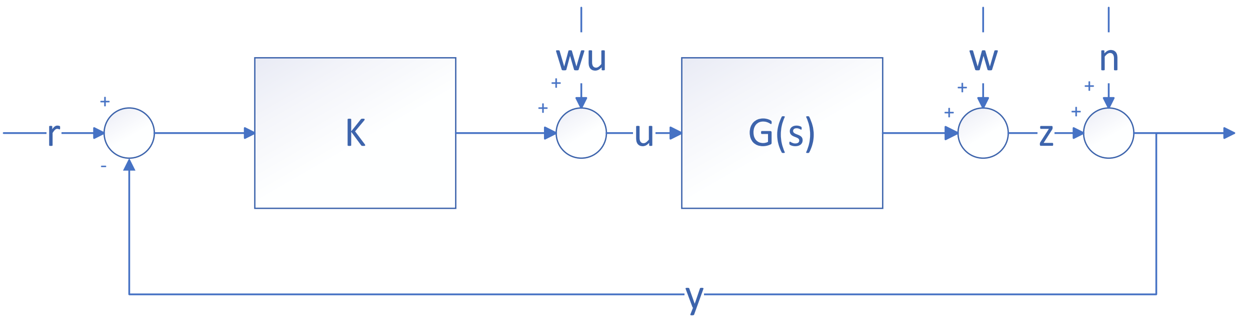
c. From the 1 DOF configuration identify the *loop transfer function*, *closed loop transfer function*, *sensitivity function* and *complementary sensitivity function*. Then answer: If the sensitivity function is known, can you calculate the loop transfer function? If the answer is yes, show the resulting formula for *L*. (2 p.)

Solution:

a.

$$\dim w = \dim n = \dim z = \dim y = \dim r = m$$
$$\dim u = \dim w_u = n$$
$$\dim G = mxn$$
$$\dim F_y = \dim F_r = nxm$$

b. *Fy* =*Fr* (=*K*). Figure: just remove the blocks *Fy* , *Fr* and set *K* in front of *G*, inside the loop. Denote the feedback signal with minus sign.



c.

$$\begin{cases} y = z + n \\ z = Gu + w \\ u = Kr - Ky + w_u \end{cases}$$

$$u = Kr - Kz - Kn + w_u$$

$$z = GKr - GKz - GKn + Gw_u + w$$

$$z(I + GK) = GKr - GKn + Gw_u + w$$

$$z = \underbrace{(I+GK)^{-1}GK}_{G_c}\, r - \underbrace{(I+GK)^{-1}GK}_{T}\, n + \underset{L}{(I+GK)^{-1}}Gw_u + \underbrace{(I+GK)^{-1}}_{S}\, w$$

$$L = GK$$

$$G_c = (I+L)^{-1}L$$

$$S = (I+L)^{-1}$$

$$T = (I+L)^{-1}L = G_c \quad \text{(1 DOF)}$$

$$S = (I+L)^{-1} \Rightarrow I + L = S^{-1} \Rightarrow L = S^{-1} - I$$

3. **a**. Let G and F_y be matrices of dimensions m x n and n x m respectively (m and n are positive integers). Calculate and try to get as simple result as possible to

$$(I + GF_y)^{-1}GF_y - GF_y(I + GF_y)^{-1} = ?$$

where the inverse matrices are assumed to exist and I:s are identity matrices of appropriate dimensions. You may use a well-known matrix identity without proving it. (2 p.)

b. Consider again the figure in Problem 2. and let the control and load disturbances be zero. Show and discuss the meaning of the below formula from control viewpoint (note in deriving the formula: MIMO case)

$$\mathrm{u} = \mathrm{G}^{-1}\left[\mathrm{G_c r} - (1 - \mathrm{S})\mathrm{n}\right] \qquad \text{(3 p.)}$$

Solution:

a. The two expressions are equal, and therefore the result is the zero matrix $0_m = 0_{mxm}$. To see that use the push-through rule

$$(I+GF_y)^{-1}GF_y = G(I+F_yG)^{-1}F_y = GF_y(I+GF_y)^{-1}$$

A direct way would be to take $A = GF_y$, $B=I_m$ in

$$(I+AB)^{-1}A = A(I+BA)^{-1}$$

b. From the figure

$$\begin{aligned}
&u = F_r r - F_y(Gu+n) = F_r r - F_y G u - F_y n \\
&\Rightarrow (I+F_yG)u = F_r r - F_y n \\
&\Rightarrow u = (I+F_yG)^{-1}F_r r - (I+F_yG)^{-1}F_y n \\
&\Rightarrow u = G^{-1}G(I+F_yG)^{-1}F_r r - G^{-1}G(I+F_yG)^{-1}F_y n \\
&\Rightarrow u = G^{-1}(I+GF_y)^{-1}GF_r r - G^{-1}(I+GF_y)^{-1}GF_y n \\
&\Rightarrow u = G^{-1}(G_c r - Tn) \\
&\Rightarrow u = G^{-1}[G_c r - (1-S)n]
\end{aligned}$$

where the push-through rule has been effectively used. The result shows that the process with disturbance is controlled as

$$y = Gu + n = G_c r - (1-S)n + n = G_c r + Sn = G_c r \approx r$$

when the sensitivity function *S* is zero and the closed loop transfer function, approximately *T*, is identity. Then the disturbance *n* would be fully compensated. Note that the same would be obtained by the control law

$$u = G^{-1}(r-n)$$

but that would need the disturbance *n* to be measured. Without that the same result can be obtained by loop shaping, at least approximately adjusting

$$S \approx 0, \Rightarrow T \approx G_c \approx I$$

Note that the result also means that in control we are facing the fact of finding at least an approximative inverse of the process transfer function, even if that is not usually visible explicitly in control design methods.