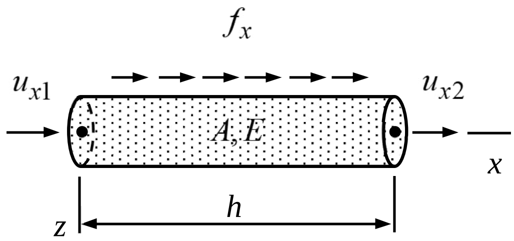
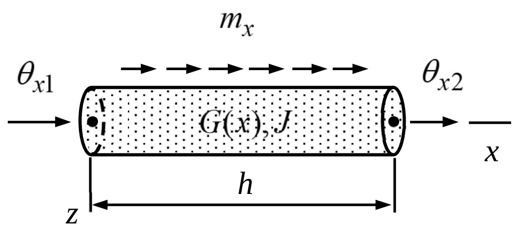
MEC-E1050 Finite Element Method in Solids, week 46/2023

1. Consider a bar element when A and E are constants and $f_x = (1 - x/h) f_{x1} + (x/h) f_{x2}$ is the linear distributed force. Derive the virtual work expression of linear bar element. Use the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/h)u_{x1} + (x/h)u_{x2}$.



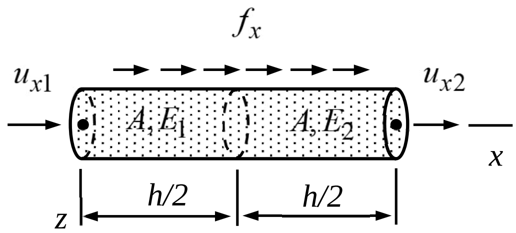
Answer $\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix})$

2. Derive the virtual work expression of a torsion bar, when J and m_x are constants and shear modulus G is linear and defined by the nodal values G_1 and G_2. Use approximation $\phi = (1 - x/h)\theta_{x1} + (x/h)\theta_{x2}$. Virtual work density of the torsion bar model is $\delta w_\Omega = -(d\delta\phi/dx)GJ(d\phi/dx) + \delta\phi m_x$.



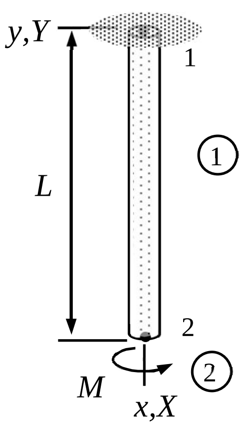
Answer $\delta W = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{G_1 + G_2}{2}\frac{J}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$

3. Consider a bar element having constant A and f_x and a piecewise constant E as shown in the figure. Derive the virtual work expression of the element by using the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ of the bar model and interpolant $u = (1 - x/h)u_{x1} + (x/h)u_{x2}$ to the nodal displacements.



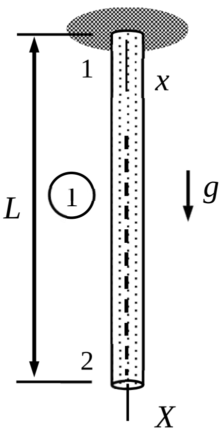
Answer $\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{E_1 + E_2}{2}\frac{A}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$

4. Consider the torsion bar (1) of the figure loaded by torque M (2) acting on the free end. Determine the rotation θ_{X2} at the free end, if the polar moment J is constant and shear modulus G varies linearly so that the values at the nodes are G_1 and G_2. Start with the virtual work density $\delta w_\Omega = -(d\delta\phi/dx)GJ(d\phi/dx) + \delta\phi m_x$ and use linear approximation to rotation (a linear two-node element).



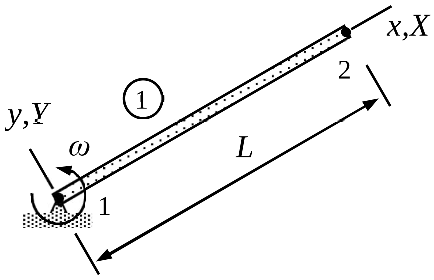
Answer $\theta_{X2} = -2\frac{ML}{(G_1 + G_2)J}$

5. Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/L)u_{x1} + (x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



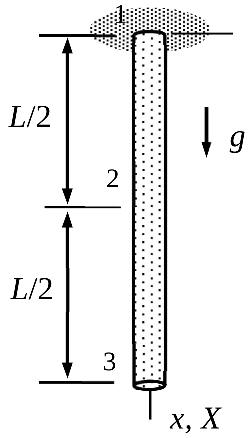
Answer $u_{X2} = \dfrac{\rho g L^2}{2E}$

6. Structural coordinate system and the bar shown are rotating in a plane with a constant angular speed $\omega_Z = \omega$. Material properties E, ρ and the cross-sectional area A are constants. Determine the nodal displacement u_{X2} at the free end using just one linear element. The *volume force* due to the rotation is given by $\vec{f} = -\rho\vec{a} = -\rho\vec{\omega}\times(\vec{\omega}\times\vec{r})$ in which $\vec{\omega} = \omega\vec{k}$ and $\vec{r} = x\vec{i}$.



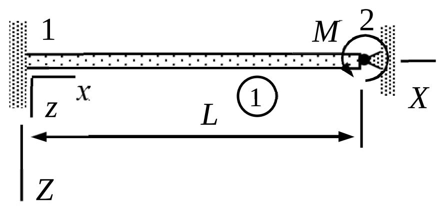
Answer $u_{X2} = \dfrac{1}{3}\dfrac{\rho\omega^2 L^3}{E}$

7. Consider the bar of the figure loaded by its own weight. Determine the displacement of the free end with one element. Use virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and quadratic approximation $u = (1 - 3\xi + 2\xi^2)u_{x1} + 4\xi(1-\xi)u_{x2} + \xi(2\xi - 1)u_{x3}$ in which $\xi = x/L$. Cross-sectional area of the bar A, acceleration by gravity g, and material properties E and ρ are constants.



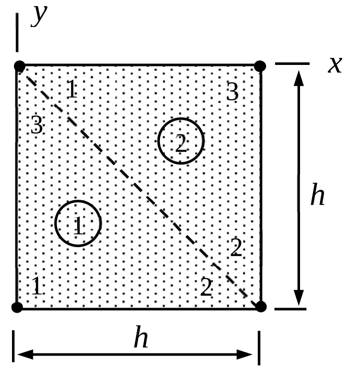
Answer $u_{X3} = \dfrac{1}{2}\dfrac{\rho g L^2}{E}$

8. Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_\Omega = -(d^2\delta w/dx^2)EI_{yy}(d^2w/dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



Answer $\theta_{Y2} = \dfrac{1}{4}\dfrac{ML}{EI}$

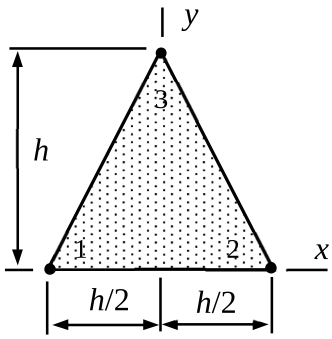
9. Deduce the shape functions of the triangle elements 1 and 2 shown in the figure in terms of the (material) xy – coordinates.



Answer $\mathbf{N}^1 = \frac{1}{h}\begin{Bmatrix} -y-x \\ x \\ h+y \end{Bmatrix}$ and $\mathbf{N}^2 = \frac{1}{h}\begin{Bmatrix} h-x \\ -y \\ x+y \end{Bmatrix}$

10. Using a linear interpolant to the nodal values, determine

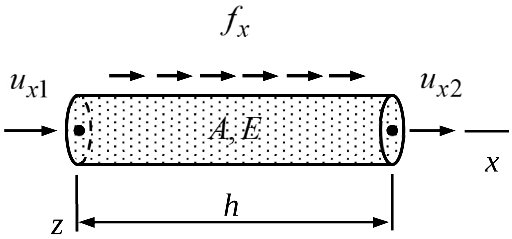
$$u_x, \ \frac{\partial u_x}{\partial x}, \ \frac{\partial u_x}{\partial y}, \ \text{and} \ I = \int_{\Omega^e} u_x d\Omega,$$



for the element shown. The nodal values of the displacement component $u_x(x, y)$ are $u_{x1} = \mathrm{a}$, $u_{x2} = -\mathrm{a}$, and $u_{x3} = 2\mathrm{a}$.

Answer $u_x = 2\frac{\mathrm{a}}{h}(y-x)$, $\frac{\partial u_x}{\partial x} = -2\frac{\mathrm{a}}{h}$, $\frac{\partial u_x}{\partial y} = 2\frac{\mathrm{a}}{h}$, $I = \frac{1}{3}\mathrm{a}h^2$

Consider a bar element when A and E are constants and $f_x = (1 - x/h) f_{x1} + (x/h) f_{x2}$ is the linear distributed force. Derive the virtual work expression of linear bar element. Use the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/h)u_{x1} + (x/h)u_{x2}$.



Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are first substituted into the density expression which is followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x \quad \text{and} \quad u = (1 - \frac{x}{h})u_{x1} + \frac{x}{h}u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u = \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \delta u = \begin{Bmatrix} 1 - x/h \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix},$$

$$\frac{du}{dx} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \frac{d\delta u}{dx} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}.$$

When the approximation is substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\mathrm{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA/h^2 & -EA/h^2 \\ -EA/h^2 & EA/h^2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta w_\Omega^{\mathrm{ext}} = \delta u f_x = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} (1 - x/h)^2 & (1 - x/h)(x/h) \\ (1 - x/h)(x/h) & (x/h)^2 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}.$$

Integration over the element gives the virtual work expressions of the internal and external forces

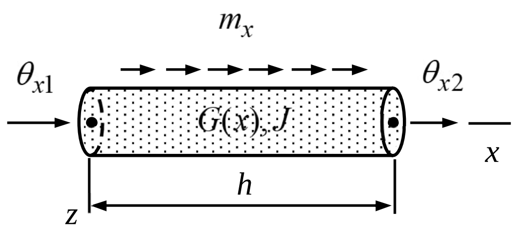
$$\delta W^{\mathrm{int}} = \int_0^h \delta w_\Omega^{\mathrm{int}} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA/h & -EA/h \\ -EA/h & EA/h \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\mathrm{ext}} = \int_0^h \delta w_\Omega^{\mathrm{ext}} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 2h/6 & h/6 \\ h/6 & 2h/6 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}.$$

Virtual work expression of bar element is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \left(\begin{bmatrix} EA/h & -EA/h \\ -EA/h & EA/h \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{bmatrix} 2h/6 & h/6 \\ h/6 & 2h/6 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \right).$$

Derive the virtual work expression of a torsion bar, when J and m_x are constants and shear modulus G is linear and defined by the nodal values G_1 and G_2. Use approximation $\phi = (1 - x/h)\theta_{x1} + (x/h)\theta_{x2}$. Virtual work density of the torsion bar model is $\delta w_\Omega = -\delta(d\phi/dx)GJ(d\phi/dx) + \delta\phi m_x$.



Solution

Virtual work density of the torsion bar model

$$\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} + \delta\phi m_x$$

depends on the polar moment of area J, shear modulus G, and moment per unit length m_x. Expression is valid also when data are not constant. Virtual work expression (element contribution)

$$\delta W = \int_\Omega \delta w_\Omega d\Omega$$

is integral of the density over the domain $\Omega =]0, h[$ occupied by the element.

Assuming that the origin of the material coordinate system is placed at node 1, linear approximation (and its derivatives as well as their variations) to the axial rotation ϕ take the forms

$$\phi = \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \; \frac{d\phi}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \; \delta\phi = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}, \; \frac{d\delta\phi}{dx} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

As the shear modulus of the material is known to be linear and defined by its nodal values

$$G = \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix} = (1 - \frac{x}{h})G_1 + \frac{x}{h}G_2.$$

When the approximation to ϕ and expression for G are substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} GJ \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GJ}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} [(1 - \frac{x}{h})G_1 + \frac{x}{h}G_2]\frac{J}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

and

$$\delta w_{\Omega}^{\text{ext}} = \delta\phi m_x \quad \Rightarrow \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix} m_x \,.$$

Virtual work expressions are integrals of the virtual work densities over the domain Ω occupied by the element (here $\Omega =]0, h[$)

$$\delta W^{\text{int}} = \int_0^h \delta w_{\Omega}^{\text{int}} dx \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\int_0^h \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} [(1-\frac{x}{h})G_1 + \frac{x}{h}G_2]\frac{J}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} dx \quad \Leftrightarrow$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{G_1 + G_2}{2}\frac{J}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}.$$

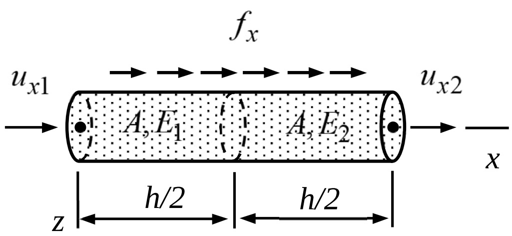
$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \Rightarrow$$

$$\delta W^{\text{ext}} = \int_0^h \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix} m_x dx = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Virtual work expression of the element is the sum of internal and external parts

$$\delta W = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{G_1 + G_2}{2}\frac{J}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) \,.$$

Consider a bar element having constant A and f_x and a piecewise constant E as shown in the figure. Derive the virtual work expression of the element by using the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ of the bar model and interpolant $u = (1-x/h)u_{x1} + (x/h)u_{x2}$ to the nodal displacements.



Solution

Virtual work densities of the internal and external distributed forces of the bar model

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u f_x$$

depend on the cross-sectional area A, Young's modulus E, and force per unit length f_x. Expressions are valid also when the data (A, E, f_x) are not constants. Virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

are integrals of the densities over the domain $\Omega =]0,h[$. As a virtual work expression of the bar element with a varying Young's modulus is not available in formulae collection, it needs to be calculated from scratch. Here

$$E = \begin{cases} E_1 & 0 \le x \le h/2 \\ E_2 & h/2 < x \le h \end{cases}.$$

Assuming that the origin of the material coordinate system is placed at node 1, linear approximation (and its derivatives as well as their variations) to the axial displacement ϕ and Young's modulus E are given by

$$u = \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \text{so}$$

$$\delta u = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix} \quad \text{and} \quad \frac{d\delta u}{dx} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

When the approximation to u and the expression of E are substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E_1 A \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad 0 \le x \le \frac{h}{2},$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E_2 A \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \frac{h}{2} < x \le h \,.$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} f_x \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}.$$

Virtual work expressions are integrals of the virtual work densities over the domain Ω occupied by the element. Here $\Omega =]0, h[$ needs to be divided into two parts since the integrand is piecewise constant

$$\delta W^{\text{int}} = \int_0^h \delta w_{\Omega}^{\text{int}} dx = \int_0^{h/2} \delta w_{\Omega}^{\text{int}} dx + \int_{h/2}^{h} \delta w_{\Omega}^{\text{int}} dx \quad \text{in which}$$

$$\int_0^{h/2} \delta w_{\Omega}^{\text{int}} dx = -\int_0^{h/2} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\int_{h/2}^{h} \delta w_{\Omega}^{\text{int}} dx = -\int_{h/2}^{h} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}.$$

and

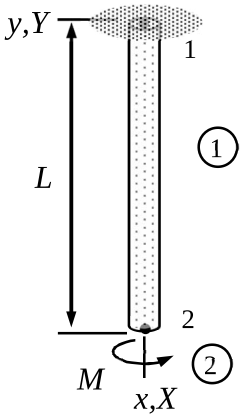
$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \Rightarrow \quad \delta W^{\text{ext}} = \int_0^h \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} f_x \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Virtual work expression of the element is the sum of internal and external parts

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} + \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E_1 + E_2}{2} \frac{A}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right). \quad \leftarrow$$

Consider the torsion bar (1) of the figure loaded by torque M (2) acting on the free end. Determine the rotation θ_{X2} at the free end, if the polar moment J is constant and shear modulus G varies linearly so that the values at the nodes are G_1 and G_2. Start with the virtual work density $\delta w_\Omega = -\delta(d\phi/dx)GJ(d\phi/dx) + \delta\phi m_x$ and use linear approximation to rotation (a linear two-node element).



Solution

Virtual work densities of the internal and external distributed forces of the torsion bar model

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta\phi m_x$$

depend on the polar moment J, shear modulus G, and moment per unit length m_x. Virtual work expression is obtained as integrals

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} \quad \text{where} \quad \delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

over the element. In this case $\Omega =]0, L[$ (just one element of length L) and $d\Omega = dx$. As "ready-to-use" virtual work expression of a torsion bar with varying shear modulus is not available in the formulae collection of the course, it needs to be calculated by using the virtual work density and approximation/interpolant to rotation.

Assuming that the origin of the material coordinate system is placed at node 1 and coincides with the structural system (for simplicity), linear approximation to the axial rotation ϕ and the expression of the shear modulus G are

$$\phi = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix} = \frac{x}{L}\theta_{X2} \text{ and } G = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix} = (1 - \frac{x}{L})G_1 + \frac{x}{L}G_2 .$$

When the approximation to ϕ and the expression of the shear modulus G are substituted there, virtual work density of internal forces becomes (external part coming from the distributed moment vanishes here)

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} = -\frac{1}{L}\delta\theta_{X2}[(1-\frac{x}{L})G_1 + \frac{x}{L}G_2]J\frac{1}{L}\theta_{X2} .$$

Virtual work expression is the integral of virtual work density over domain Ω occupied by element (here $\Omega =]0, L[$)

$$\delta W^{\text{int}} = \int_0^L \delta w_\Omega^{\text{int}} dx = -\frac{1}{L}\delta\theta_{X2}(\frac{L}{2}G_1 + \frac{L}{2}G_2)J\frac{1}{L}\theta_{X2} = -\delta\theta_{X2}\frac{G_1 + G_2}{2}\frac{J}{L}\theta_{X2} \quad \Rightarrow$$

$$\delta W^1 = -\delta\theta_{X2}\frac{G_1+G_2}{2}\frac{J}{L}\theta_{X2}.$$

Virtual work expression of the point force/moment is available in the formulae collection (the definition of work can also be used)

$$\delta W^{\text{ext}} = \begin{Bmatrix}\delta u_{Xi}\\ \delta u_{Yi}\\ \delta u_{Zi}\end{Bmatrix}^{\text{T}}\begin{Bmatrix}F_{Xi}\\ F_{Yi}\\ F_{Zi}\end{Bmatrix}+\begin{Bmatrix}\delta\theta_{Xi}\\ \delta\theta_{Yi}\\ \delta\theta_{Zi}\end{Bmatrix}^{\text{T}}\begin{Bmatrix}M_{Xi}\\ M_{Yi}\\ M_{Zi}\end{Bmatrix} = -\delta\theta_{X2}M \quad\Rightarrow\quad \delta W^2 = -\delta\theta_{X2}M\,.$$

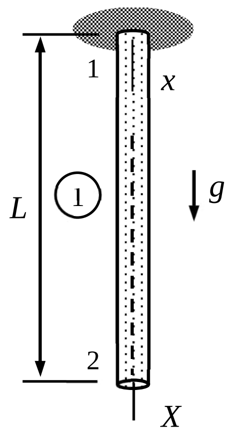
Virtual work expression of the structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{X2}(\frac{G_1+G_2}{2}\frac{J}{L}\theta_{X2}+M).$$

Principle of virtual work $\delta W = 0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\text{T}}\mathbf{F} = 0\ \forall\delta\mathbf{a} \ \Leftrightarrow\ \mathbf{F} = 0$ imply

$$\frac{G_1+G_2}{2}\frac{J}{L}\theta_{X2} - (-M) = 0 \quad\Leftrightarrow\quad \theta_{X2} = -2\frac{ML}{(G_1+G_2)J}\,.$$

Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx)+\delta u f_x$ and approximation $u=(1-x/L)u_{x1}+(x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g and material properties E and ρ are constants.



Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are just substituted into the density expression followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x \quad \text{and} \quad u=(1-\frac{x}{L})u_{x1}+\frac{x}{L}u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \quad\Rightarrow\quad \delta u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix},$$

$$\frac{du}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \quad\Rightarrow\quad \frac{d\delta u}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

When the approximation is substituted there, virtual work density expression of the bar model takes the form

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x = -\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}+\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}-1\\ 1\end{Bmatrix}\frac{1}{L}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f) \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f)\ .$$

Finally, integration over the element gives the virtual work expression of the bar element

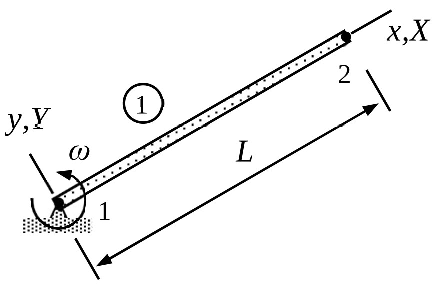
$$\delta W = \int_0^L \delta w_\Omega dx = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{fL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}).$$

Finding the displacement of the free end follows the usual lines. Here, $f_x = \rho g A$, $u_{x1} = u_{X1} = 0$, and $u_{x2} = u_{X2}$

$$\delta W = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\rho g A L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) = -\delta u_{X2}\left(\frac{EA}{L}u_{X2} - \frac{\rho g A L}{2}\right) = 0 \quad \forall \delta u_{X2} \quad \Leftrightarrow$$

$$\frac{EA}{L}u_{X2} - \frac{\rho g A L}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\rho g L^2}{2E}.$$

Structural coordinate system and the bar shown are rotating in a plane with a constant angular speed $\omega_Z = \omega$. Material properties E, ρ and the cross-sectional area A are constants. Determine the nodal displacement u_{X2} at the free end using just one linear element. The *volume force* due to the rotation is given by $\vec{f} = -\rho\vec{a} = -\rho\vec{\omega}\times(\vec{\omega}\times\vec{r})$ in which $\vec{\omega} = \omega\vec{k}$ and $\vec{r} = x\vec{i}$.



Solution

Virtual work densities of the internal and external distributed forces of the bar model

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx} + \delta u f_x \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u f_x$$

depend on the cross-sectional area A, Young's modulus E, and external force per unit length f_x. Virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

are integrals of the densities over the domain $\Omega =]0,h[$ occupied by the element. Principle of virtual work corresponds to equilibrium equations that are valid in their simple form ($\vec{F} = m\vec{a}$ with $\vec{a} = 0$ etc.) in an inertial frame of reference. The correct form for a coordinate system which rotates with a constant angular speed $\vec{\omega}$ in the manner shown in the figure ($\vec{F} = m[\vec{a}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r})]$ with $\vec{a}_r = 0$) gives rise to a force per unit volume in the same manner as gravity is acting on the body. In the present case, (material system and structural system coincide) force per unit length in the direction of the $x-$ axis takes the form

$$f_x = \vec{f}\cdot\vec{i} = -\rho A[\omega\vec{k}\times(\omega\vec{k}\times x\vec{i})]\cdot\vec{i} = \rho A\omega^2 x .$$

Virtual work expression of bar element with a varying distributed external force is not available in formulae collection and it needs to be calculated from scratch. According to the formulae collection

$$u = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\text{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \text{in which} \quad \xi = \frac{x}{h} .$$

Assuming that the origin of the material coordinate system is placed at node 1 and coincides with the structural system, approximation to the axial displacement and its derivatives and variations are (In hand calculations, it is advantageous to use information about boundary conditions etc. as soon as possible.)

$$u = \frac{1}{L}\begin{Bmatrix} L-x \\ x \end{Bmatrix}^{\text{T}} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = \frac{x}{L}u_{X2} \quad \text{and} \quad \delta u = \frac{x}{L}\delta u_{X2} ,$$

$$\frac{du}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\u_{X2}\end{Bmatrix}=\frac{1}{L}u_{X2} \quad \text{and} \quad \delta\frac{du}{dx}=\frac{1}{L}\delta u_{X2}.$$

When the approximation to u is substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_{\Omega}^{\mathrm{int}}=-\frac{1}{L}\delta u_{X2}EA\frac{1}{L}u_{X2} \quad \text{and} \quad \delta w_{\Omega}^{\mathrm{ext}}=\frac{x}{L}\delta u_{X2}\rho A\omega^{2}x.$$

Virtual work expressions are integrals of the densities over the domain Ω occupied by the element

$$\delta W^{\mathrm{int}}=\int_{0}^{L}\delta w_{\Omega}^{\mathrm{int}}dx=-\delta u_{X2}\frac{EA}{L}u_{X2},$$

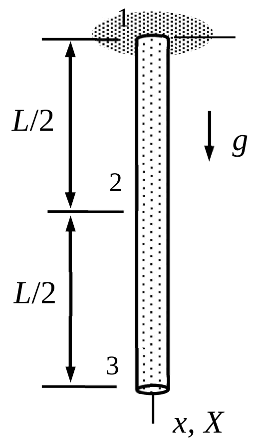
$$\delta W^{\mathrm{ext}}=\int_{0}^{L}\delta W^{\mathrm{int}}dx=\delta u_{X2}\frac{1}{3}L^{2}\rho A\omega^{2}.$$

Virtual work expression of the element is sum of the internal and external parts

$$\delta W=\delta W^{\mathrm{int}}+\delta W^{\mathrm{ext}}=-\delta u_{X2}(\frac{EA}{L}u_{X2}-\frac{1}{3}L^{2}\rho A\omega^{2}).$$

Principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\ \ \forall\delta\mathbf{a}\ \Leftrightarrow\ \mathbf{R}=0$ imply

$$\frac{EA}{L}u_{X2}-\frac{1}{3}L^{2}\rho A\omega^{2}=0 \quad \Leftrightarrow \quad u_{X2}=\frac{1}{3}\frac{L^{3}\rho\omega^{2}}{E}. \ \leftarrow$$



Consider the bar of the figure loaded by its own weight. Determine the displacement of the free end with one element. Use virtual work density expression $\delta w_{\Omega} = -\delta(du/dx)EA(du/dx) + \delta u f_x$ and quadratic approximation $u = (1-3\xi+2\xi^2)u_{x1} + 4\xi(1-\xi)u_{x2} + \xi(2\xi-1)u_{x3}$ in which $\xi = x/L$. Cross-sectional area of the bar A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

Virtual work densities of the internal and external distributed forces of the bar model

$$\delta w_{\Omega}^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}} = \delta u f_x$$

depend on the cross-sectional area A, Young's modulus E, and force per unit length f_x. Virtual work expressions

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega$$

are obtained as integrals over the domain $\Omega =]0, h[$ occupied by the element (here $h = L$).

Virtual work expression of the bar element, when approximation is quadratic (a three-node element), is not available in formulae collection and it needs to be calculated from scratch. In hand calculations, it is advantageous to use information about boundary conditions etc. as soon as possible. In the present case, force per unit length is due to the weight

$$f_x = \rho g A .$$

Assuming that the origin of the material coordinate system is placed at node 1 and coincides with the structural system, quadratic approximation to the axial displacement u (see the formula collection) and its derivatives and variations are given by

$$u = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \\ u_{X3} \end{Bmatrix} = \begin{Bmatrix} 4\frac{x}{L} - 4(\frac{x}{L})^2 \\ 2(\frac{x}{L})^2 - \frac{x}{L} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} \quad \Rightarrow$$

$$\delta u = \begin{Bmatrix} 4\frac{x}{L} - 4(\frac{x}{L})^2 \\ 2(\frac{x}{L})^2 - \frac{x}{L} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}, \quad \frac{du}{dx} = \frac{1}{L} \begin{Bmatrix} 4-8\frac{x}{L} \\ 4\frac{x}{L}-1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix}, \text{ and } \frac{d\delta u}{dx} = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L} \begin{Bmatrix} 4-8\frac{x}{L} \\ 4\frac{x}{L}-1 \end{Bmatrix}.$$

When the approximation to u is substituted there, virtual work densities of the internal and external forces become

$$\delta w_{\Omega}^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \begin{Bmatrix} 4L-8x \\ 4x-L \end{Bmatrix} EA \frac{1}{L^2} \begin{Bmatrix} 4L-8x \\ 4x-L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L^4} \begin{bmatrix} (4L-8x)^2 & (4L-8x)(4x-L) \\ (4x-L)(4L-8x) & (4x-L)^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \delta u f_x = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 4x/L-4(x/L)^2 \\ 2(x/L)^2-x/L \end{Bmatrix} \rho g A.$$

Virtual work expressions are integrals of the virtual work densities over the domain Ω occupied by the element

$$\delta W^{\text{int}} = \int_0^L \delta w_{\Omega}^{\text{int}} dx \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L^4} \int_0^L \begin{bmatrix} (4L-8x)^2 & (4L-8x)(4x-L) \\ (4x-L)(4L-8x) & (4x-L)^2 \end{bmatrix} dx \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \Rightarrow$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \int_0^L \frac{1}{L^2} \begin{Bmatrix} 4xL-4x^2 \\ 2x^2-xL \end{Bmatrix} dx \rho g A = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g A L}{6} \begin{Bmatrix} 4 \\ 1 \end{Bmatrix}.$$

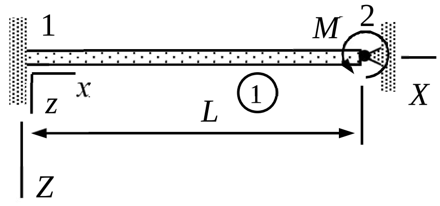
Virtual work expression of the element is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g A L}{6} \begin{Bmatrix} 4 \\ 1 \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \ \ \forall \delta \mathbf{a} \ \Leftrightarrow \ \mathbf{R} = 0$ imply

$$\frac{EA}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g A L}{6} \begin{Bmatrix} 4 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \frac{\rho g L^2}{6E} \frac{1}{48} \begin{bmatrix} 7 & 8 \\ 8 & 16 \end{bmatrix} \begin{Bmatrix} 4 \\ 1 \end{Bmatrix} = \frac{1}{2} \frac{\rho g L^2}{E} \begin{Bmatrix} 3/4 \\ 1 \end{Bmatrix}. \ \leftarrow$$

Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_{\Omega} = -\delta(d^2 w / dx^2) EI_{yy} (d^2 w / dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



Solution

In the xz – plane problem bending problem, when x-axis is chosen to coincide with the neutral axis, virtual work densities of the beam bending mode are

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2 \delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

Approximation is the first thing to be considered. The left end of the beam is clamped and the right end support does not allow transverse displacement. As only $\theta_{y2} = \theta_{Y2}$ is non-zero, approximation to w simplifies into the form (see the formulae collection for the cubic beam bending approximation)

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2 \xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = L(\frac{x}{L})^2(1-\frac{x}{L})\theta_{Y2} \quad \Rightarrow \quad \frac{d^2 w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\theta_{Y2} \text{ and}$$

$$\delta w = L(\frac{x}{L})^2(1-\frac{x}{L})\delta\theta_{Y2} \quad \Rightarrow \quad \frac{d^2 \delta w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\delta\theta_{Y2} .$$

When the approximation is substituted there, virtual work density takes the form (external distributed force vanishes)

$$\delta w_{\Omega} = -\frac{d^2 \delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} = -\delta\theta_{Y2} \frac{EI}{L^4} \theta_{Y2} (2L-6x)^2 .$$

Integration over the domain $\Omega =]0, L[$ gives the virtual work expressions (beam is considered as element 1)

$$\delta W^1 = \int_0^L \delta w_{\Omega} dx = -\delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2} .$$

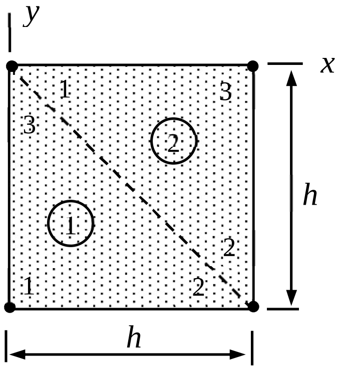
The external moment gives the contribution (element 2)

$$\delta W^2 = \delta\theta_{Y2} M .$$

Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} - M) = 0 \quad \forall \delta u_{Z2} \quad \Leftrightarrow \quad 4\frac{EI}{L}\theta_{Y2} - M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{4}\frac{ML}{EI} \ . \quad \leftarrow$$

Derive the shape functions of the triangle elements 1 and 2 shown in the figure in terms of the (material) xy – coordinates.



Solution

Shape functions of the linear triangle element are given by the simple formula

$$\mathbf{N}=\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}$$

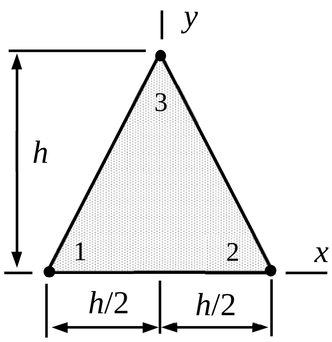
in which the subscripts refer to coordinates of the three nodes. Order of the node numbers does not matter:

$$\mathbf{N}^1=\begin{bmatrix}1 & 1 & 1\\ 0 & h & 0\\ -h & -h & 0\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}=\frac{1}{h^2}\begin{bmatrix}0 & 0 & h^2\\ -h & h & 0\\ -h & 0 & h\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}=\frac{1}{h}\begin{bmatrix}0 & -1 & -1\\ 0 & 1 & 0\\ h & 0 & 1\end{bmatrix}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}=\frac{1}{h}\begin{Bmatrix}-y-x\\ x\\ h+y\end{Bmatrix},$$

$$\mathbf{N}^2=\begin{bmatrix}1 & 1 & 1\\ 0 & h & h\\ 0 & -h & 0\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}=\frac{1}{h^2}\begin{bmatrix}h^2 & 0 & 0\\ -h & 0 & h\\ 0 & -h & h\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}=\frac{1}{h}\begin{bmatrix}h & -1 & 0\\ 0 & 0 & -1\\ 0 & 1 & 1\end{bmatrix}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}=\frac{1}{h}\begin{Bmatrix}h-x\\ -y\\ x+y\end{Bmatrix}.$$

Using a linear interpolant to the nodal values, determine

$$u_x, \ \frac{\partial u_x}{\partial x}, \ \frac{\partial u_x}{\partial y}, \ \text{and } I = \int_{\Omega^e} u_x d\Omega,$$



for the element shown. The nodal values of the displacement component $u_x(x, y)$ are $u_{x1} = \mathrm{a}$, $u_{x2} = -\mathrm{a}$, and $u_{x3} = 2\mathrm{a}$.

Solution

The shape functions of a three-node triangular element in xy – coordinates are linear and therefore of the form

$$N_i = a_i + b_i x + c_i y = \{1 \quad x \quad y\}\begin{Bmatrix} a_i \\ b_i \\ c_i \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as the basis functions should take the value 1 at their own nodes and vanish at all the other nodes. The closed form solution to the shape functions is given by

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{\det}\begin{Bmatrix} x_3(y - y_2) + x(y_2 - y_3) + x_2(-y + y_3) \\ x_3(-y + y_1) + x_1(y - y_3) + x(-y_1 + y_3) \\ x_2(y - y_1) + x(y_1 - y_2) + x_1(-y + y_2) \end{Bmatrix}$$

where $\det = -x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3$. Often, the linear shape functions can be deduced directly from a figure. However, the generic expressions work also when intuition does not:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -h/2 \\ h/2 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ h \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \frac{1}{2h}\begin{Bmatrix} h - 2x - y \\ h + 2x - y \\ 2y \end{Bmatrix}.$$

With the given nodal values, element interpolant (approximation) becomes

$$u_x = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} = \left(\frac{1}{2} - \frac{x}{h} - \frac{y}{2h}\right)\mathrm{a} + \left(\frac{1}{2} + \frac{x}{h} - \frac{y}{2h}\right)(-\mathrm{a}) + \frac{y}{h}2\mathrm{a} = 2\frac{\mathrm{a}}{h}(y - x).$$

Thus

$$\frac{\partial u_x}{\partial x} = -2\frac{\mathrm{a}}{h},$$

$$\frac{\partial u_x}{\partial y} = 2\frac{\mathrm{a}}{h},$$

$$\int_{\Omega^e} u_x d\Omega = \int_0^h \left[\int_{(y-h)/2}^{(h-y)/2} 2\frac{\mathrm{a}}{h}(y-x)dx\right]dy = \int_0^h 2\frac{\mathrm{a}}{h} y(h-y)dy = \frac{1}{3}\mathrm{a}h^2 .$$