Problem 10.1: ADMM for Stochastic Linear Optimization Problems

Consider the following two-stage stochastic linear optimization problem

$$\zeta = \min_{x} \ \left\{ c^\top x + \mathcal{Q}(x) : x \in X \right\}, \tag{1}$$

with the variables $x \in \mathbb{R}^{n_x}$ and known first-stage costs $c \in \mathbb{R}^{n_x}$. The set X consists of linear constraints on the variables x. The function $\mathcal{Q} : \mathbb{R}^{n_x} \to \mathbb{R}$ is the *expected recourse value*

$$\mathcal{Q}(x) = \mathbf{E}_\xi \left[\min_{y} \ \left\{ q(\xi)^\top y : W(\xi) y = h(\xi) - T(\xi) x, \ y \in Y(\xi) \right\} \right] \tag{2}$$

with variables $y \in \mathbb{R}^{n_y}$. Values of the vectors $q(\xi) \in \mathbb{R}^{n_y}, h(\xi) \in \mathbb{R}^n$; matrices $W(\xi) \in \mathbb{R}^{n \times n_y}$, $T(\xi) \in \mathbb{R}^{n \times n_x}$; and the set $Y(\xi)$ all depend on realizations of a random variable ξ.

Suppose that ξ is associated with a discrete distribution indexed by a finite set $\mathcal{S}$, consisting of realizations $\xi_1, \dots, \xi_{|S|}$, corresponding to realization probabilities $p_1, \dots, p_{|S|}$. Each realization ξ_s of ξ is called a *scenario* and encodes realizations observed by the random elements

$$(q(\xi_s), h(\xi_s), W(\xi_s), T(\xi_s), Y(\xi_s))$$

To simplify notation, we refer to this collection of random elements respectively as

$$(q_s, h_s, W_s, T_s, Y_s)$$

For each scenario $s \in S$, the set Y_s consists of linear constraints on the variables $y_s \in \mathbb{R}^{n_y}$. We can reformulate problem (1) as an *equivalent deterministic problem*

$$\zeta = \min_{x,y} \ \left\{ c^\top x + \sum_{s \in \mathcal{S}} p_s q_s^\top y_s : (x, y_s) \in K_s, \ \forall s \in S \right\}, \tag{3}$$

where

$$K_s = \left\{ (x, y_s) : W_s y_s = h_s - T_s x, \ x \in X, y_s \in Y_s \right\}.$$

Problem (3) has a decomposable structure that can be exploited. To induce this structure, let us introduce scenario-dependent copy variables x_s of the first-stage variable x for each scenario $s \in \mathcal{S}$. Using these copy variables, we can reformulate (3) as

$$\zeta = \min_{x,y,z} \ \left\{ \sum_{s \in \mathcal{S}} p_s (c^\top x_s + q_s^\top y_s) : (x_s, y_s) \in K_s, x_s = z, \ \forall s \in \mathcal{S}, z \in \mathbb{R}^{n_x} \right\}. \tag{4}$$

The variable $z \in \mathbb{R}^{n_x}$ is a common global variable, and the constraints $x_s = z$ for all $s \in \mathcal{S}$ enforce *nonanticipativity* for the first-stage decisions: all first-stage decisions x_s must be the same (z) for each scenario $s \in \mathcal{S}$ in the final solution.

Relaxing the nonanticipativity constraints $x_s = z$ for all $s \in \mathcal{S}$ in (4) in Lagrangian fashion yields the following *augmented Lagrangian dual function*

$$\phi(\mu) = \min_{x,y,z} \sum_{s \in \mathcal{S}} \left[p_s (c^\top x_s + q_s^\top y_s) + \mu_s^\top (x_s - z) + p_s \frac{\rho}{2} ||x_s - z||_2^2 \right] \tag{5}$$

$$\text{subject to: } (x_s, y_s) \in K_s, \ \forall s \in \mathcal{S} \tag{6}$$

By defining $v_s = \mu_s / p_s$ for all $s \in \mathcal{S}$, we can rewrite (5) – (6) as

$$\phi(v) = \min_{x,y,z} \sum_{s \in \mathcal{S}} p_s L_s^\rho (x_s, y_s, z, v_s) \tag{7}$$

$$\text{subject to: } (x_s, y_s) \in K_s, \ \forall s \in \mathcal{S} \tag{8}$$

where $L_s^\rho(x_s, y_s, z, v_s)$, defined for each $s \in S$, is the *augmented Lagrangian*

$$L_s^\rho(x_s, y_s, z, v_s) = c^\top x_s + q_s^\top y_s + v_s^\top (x_s - z) + \frac{\rho}{2}||x_s - z||_2^2 \tag{9}$$

Derive the ADMM iterations for solving the problem (7) – (8) in a distributed fashion for each scenario $s \in \mathcal{S}$ separately.