Exercise sheet 10

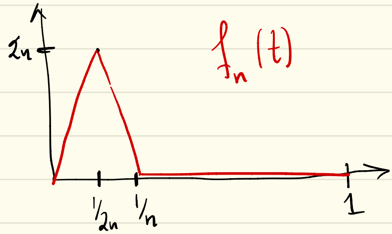
(1) Let (f_n) be the sequence of functions defined on $[0,1]$ as follows:

$$f_n(t) = \begin{cases} 4n^2 t & 0 \le t \le \frac{1}{2n} \\ 4n - 4n^2 t & \frac{1}{2n} \le t \le \frac{1}{n} \\ 0 & \frac{1}{n} \le t \le 1 \end{cases}$$

Draw a graph of $f_n(t)$, check that $\lim_{n\to\infty} f_n(x) = 0$, but

$$\lim_{n\to\infty} \int_0^1 f_n(t)\,dt \neq 0.$$

Solution:



Also since $\lim_{n\to\infty} \frac{1}{n} = 0$ we get that

$$\lim_{n\to\infty} f_n(t) = 0 \quad \text{every } t > 0.$$

Also $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$

$$\int_0^1 f_n(t)\,dt = \frac{2n \cdot \frac{1}{n}}{2} = 1 \quad \text{so}$$

$$\lim_{n\to\infty} \int_0^1 f_n(t)\,dt = 1 \neq 0.$$

(2) Verify that $\sum_{n=0}^{\infty} \frac{z^n}{1+z^n}$ converges normally in $|z|<1$. Also verify that it diverges when $|z|>1$.
(It is actually true that it diverges when $|z|\geq 1$)

Solution: Take $|z|\leq r<1$. Then $\left|\frac{z^n}{1+z^n}\right| \leq \frac{|z|^n}{1-|z|^n}$

$$\leq \frac{|z|^n}{1-r^n} \leq \frac{r^n}{1-r^n} \leq Mr^n \quad \text{where } M = \frac{1}{1-r^n}$$

Since $r<1$ we see that $\sum_{n=0}^{\infty} \frac{z^n}{1+z^n}$ converges uniformly in $|z|\leq r$ by the Weierstrass M-test. Therefore the series converges normally in $|z|<1$.
If $|z|=R>1$ then

$$\left|\frac{z^n}{1+z^n}\right| \geq \frac{|z|^n}{|z|^n \left|1+\frac{1}{z^n}\right|} = \frac{1}{\left|1+\frac{1}{z^n}\right|} \xrightarrow{\text{as } n\to\infty} 1$$

and therefore the terms in $\sum_{n=0}^{\infty} \frac{z^n}{1+z^n}$ does not converge to zero and the series diverges.

(3) Let D be a bounded domain in the complex plane. Suppose that every function in a sequence (f_n) is continuous on $\overline{D}$ and analytic in D. Assume that this sequence converges uniformly on ∂D, prove that it converges uniformly on D.

Solution: Since (f_n) converges uniformly on ∂D we know that, for each $\varepsilon > 0$ we have N so that $\sup(|f_k(z) - f_m(z)|\,;\ z \in \partial D) < \varepsilon$ when $k, m \geq N$ (the Cauchy criterion). Now

$f_k(z) - f_m(z)$ is analytic in D and continuous on $\overline{D}$.

Therefore the maximum principle tells us that

$$\sup(|f_k(z) - f_m(z)|\,;\ z \in D) = \sup(|f_k(z) - f_m(z)|\,;\ z \in \partial D)$$

Hence $\sup(|f_k(z) - f_m(z)|\,;\ z \in D) < \varepsilon$ when $k, m \geq N$ and (f_n) converges uniformly on D.

(4) Assume that (f_n) and (g_n) converges uniformly on a set A. Shows that $(f_n + g_n)$ converges uniformly on A. Also show that $(f_n g_n)$ converges uniformly if we also assume that (f_n) and (g_n) are uniformly bounded on A. (Uniformly bounded if $\exists c$ such that $\sup(|f_n(z)|\,;\ z \in A) < c$ and $\sup(|g_n(z)|\,;\ z \in A) < c$ for all n.)

[This implies that Taylor series and Laurent series can be multiplied term-by-term inside their convergence domains and yield valid series]

Solution: We use the Cauchy criterion. Let $\varepsilon > 0$.

Then there is an N so that

$|f_k(z) - f_m(z)| < \varepsilon/2$ and $|g_k(z) - g_m(z)| < \varepsilon/2$

when $k, m \geq N$.

Therefore

$$|f_k(z) + g_k(z) - f_m(z) - g_m(z)| \leq$$

$$\leq |f_k(z) - f_m(z)| + |g_k(z) - g_m(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

when $k, m \geq N$. This imply that $(f_n + g_n)$ converges uniformly.

Assume $\sup(|f_n(z)|; z \in A) < c$ and

$\sup(|g_n(z)|; z \in A) < c$.

Now choose N so that $|f_m(z) - f_k(z)| < \frac{\varepsilon}{2c}$ and $|g_m(z) - g_k(z)| < \frac{\varepsilon}{2c}$ when $m, k \geq N$.

Then $|f_m(z) g_m(z) - f_k(z) g_k(z)| =$

$$= |f_m(z) g_m(z) - f_k(z) g_m(z) + f_k(z) g_m(z) - f_k(z) g_k(z)|$$

$$\leq |g_m(z)|\,|f_m(z) - f_k(z)| + |f_k(z)|\,|g_m(z) - g_k(z)|$$

$$< c \cdot \frac{\varepsilon}{2c} + c \cdot \frac{\varepsilon}{2c} = \varepsilon \quad \text{when} \quad m, k \geq N.$$