# Exercise sheet 9

(1) A function $f$ is analytic in an annulus $D$ centered at a point $z_0$, that is $D = \{z \in \mathbb{C};\ a < |z - z_0| < b\}$, where $0 \leq a < b \leq \infty$ and $z_0 \in \mathbb{C}$. Show that

$$\int_{|z-z_0|=r} f(z)\,dz = \int_{|z-z_0|=s} f(z)\,dz \quad \text{whenever}$$

$a < r < s < b$, and $|z - z_0| = r$ and $|z - z_0| = s$ are positively oriented.

<u>Solution</u>: Let $\alpha(t) = z_0 + re^{it}$ and $\beta(t) = z_0 + se^{it}$ where $0 \leq t \leq 2\pi$. Then $n(\alpha, w) = n(\beta, w)$ for every $w \in \mathbb{C} \setminus D$ ($= 1$ when $|w - z_0| \leq a$ and $= 0$ when $|w - z_0| \geq b$)

Therefore $\sigma = (\alpha, -\beta)$ is homologous to zero in $D$ and Cauchy's Theorem says that

$$0 = \int_\sigma f(z)\,dz = \int_\alpha f(z)\,dz - \int_\beta f(z)\,dz$$

$$\Longrightarrow \int_{|z-z_0|=r} f(z)\,dz = \int_{|z-z_0|=s} f(z)\,dz$$

$\otimes$

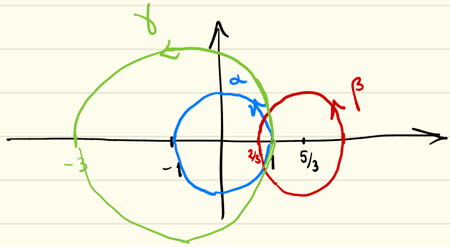
(3) Let $\alpha(t) = e^{it}$, $\beta(t) = \frac{5}{3} + e^{it}$, and $\gamma(t) = -1 + 2e^{it}$ for $0 \leq t \leq 2\pi$.

a) Is the cycle $\sigma = (\alpha, \beta, \gamma)$ homologous to zero in $\mathbb{C} \setminus \{2i, -2i\}$?

b) Are the cycles $\sigma = (\alpha, -\gamma)$ and $\tau = (\beta, \beta)$ homologous in $\mathbb{C} \setminus \overline{\Delta(0, \frac{1}{2})}$ ?

c) Are the cycles $\sigma = (\alpha, \beta, \gamma)$ and $\tau = (\gamma, \gamma, \beta)$ homologous in $\mathbb{C} \setminus \{0\}$

<u>Solution</u>:




a)

$$\left.\begin{array}{l} n(\alpha, 2i) = 0 = n(\alpha, -2i) \\ n(\beta, 2i) = 0 = n(\beta, -2i) \\ n(\gamma, 2i) = 0 = n(\gamma, -2i) \end{array}\right\}$$

since $2i$ and $-2i$ "outside" the paths
For example, $|2i - (-1)| = \sqrt{4+1} > 2$

Therefore $n(\sigma, 2i) = n(\alpha, 2i) + n(\beta, 2i) + n(\gamma, 2i) = 0$

and similarly $n(\sigma, -2i) = 0$

$\Longrightarrow$ $\sigma$ is homologous to zero in $\mathbb{C} \setminus \{2i, -2i\}$

b) For any point $z \in \overline{\Delta(0, 1/2)}$ we have $n(\alpha, z) = 1$, $n(\beta, z) = 0$, and $n(\gamma, z) = 1$

Therefore $n(\sigma, z) = n(\alpha, z) - n(\gamma, z) = 1 - 1 = 0$

and $n(\tau, z) = 2n(\beta, z) = 0$

It follows that $\sigma$ and $\tau$ are homologous in $\mathbb{C} \setminus \overline{\Delta(0, 1/2)}$.

c) We have $n(\alpha,0)=1$, $n(\beta,0)=0$, and $n(\gamma,0)=1$. Therefore $\sigma=(\alpha,\beta,\gamma)$ and $\tau=(\gamma,\gamma,\beta)$ are homologous in $\mathbb{C}\setminus\{0\}$ since

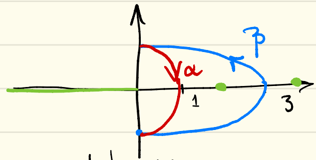
$$n(\sigma,0)=n(\alpha,0)+n(\beta,0)+n(\gamma,0)=2=$$
$$=n(\gamma,0)+n(\gamma,0)+n(\beta,0)=n(\tau,0)$$

$\otimes$

2) Let $\alpha(t)=e^{-it}$ and $\beta(t)=3\cos t+i\sin t$ for $-\pi/2\le t\le\pi/2$. Compute

$$\int_{\beta+\alpha}\frac{16\,\mathrm{Log}(z)}{z(z-2)^2(z-4)}\,dz$$

Solution:




The integrand $f(z)=\dfrac{16\,\mathrm{Log}(z)}{z(z-2)^2(z-4)}$ is analytic on $\mathbb{C}\setminus\underbrace{\big((-\infty,0]\cup\{2\}\cup\{4\}\big)}_{\text{marked green in figure}}$

Use partial fractions

$$\frac{16}{z(z-2)^2(z-4)}=\frac{A}{z}+\frac{B}{z-4}+\frac{C}{(z-2)^2}+\frac{D}{z-2}=$$

$$=\frac{A(z-4)(z-2)^2+Bz(z-2)^2+Cz(z-4)+Dz(z-4)(z-2)}{z(z-2)^2(z-4)}$$

$$=\frac{A(z-4)(z^2-4z+4)+Bz(z^2-4z+4)+Cz(z-4)+Dz(z^2-6z+8)}{z(z-2)^2(z-4)}$$

Check these as errors are likely.

Therefore $\begin{cases} -16A = 16 \\ 20A + 4B - 4C + 8D = 0 \\ -8A - 4B + C - 6D = 0 \\ A + B + D = 0 \end{cases} \Longrightarrow \begin{cases} A = -1 \\ B = 1 \\ C = -4 \\ D = 0 \end{cases}$

We get $A \int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{z} dz = 0$ since $n(\beta+\alpha, 0) = 0$

and $B \int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{z-4} dz = 0$ since $n(\beta+\alpha, 4) = 0$

We have $n(\beta+\alpha, 2) = 1$ so we calculate $C \int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{(z-2)^2} dz$ and $D \int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{z-2} dz$ using the Cauchy Integral Formula.

We get $D \int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{z-2} dz = D(2\pi i) \operatorname{Log}(2) = D\, 2\pi i \ln 2$

and, since $\frac{d}{dz} \operatorname{Log}(z) = \frac{1}{z}$,

$$C \int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{(z-2)^2} dz = \frac{C(2\pi i)}{1!} \frac{1}{2} = C\pi i$$

So, if my calculations are correct, we get

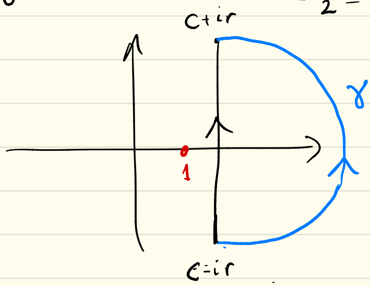
$$\int_{\beta+\alpha} \frac{16 \operatorname{Log}(z)}{z(z-2)^2(z-4)} dz = -4\pi i$$

4. Prove that $\lim\limits_{r\to\infty}\int_{c-ir}^{c+ir} \frac{1}{z\,\mathrm{Log}(z)}\,dz = 0$

when $c>1$. (Hint: Consider $\int_{\gamma} \frac{1}{z\,\mathrm{Log}(z)}\,dz$, where $\gamma(t) = c + re^{it}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.)

<u>Solution</u>:

$\frac{1}{z\,\mathrm{Log}(z)}$ is analytic when $\mathrm{Re}(z)>1$




By Cauchy's Theorem $\int_{c-ir}^{c+ir} \frac{1}{z\,\mathrm{Log}(z)}\,dz = \int_{\gamma} \frac{1}{z\,\mathrm{Log}(z)}\,dz$

On $\gamma$ we have $|z| \ge \sqrt{c^2+r^2} \ge r$ and

Notice that this estimate does not hold on $[c-ir, c+ir]$

$|\mathrm{Log}(z)| \ge \ln\sqrt{c^2+r^2}$. Therefore

$$\left|\int_{c-ir}^{c+ir} \frac{1}{z\,\mathrm{Log}(z)}\,dz\right| = \left|\int_{\gamma} \frac{1}{z\,\mathrm{Log}(z)}\,dz\right| \le \frac{\pi r}{r\ln\sqrt{c^2+r^2}} \xrightarrow[r\to\infty]{\text{when}} 0$$

We get $\lim\limits_{r\to\infty}\int_{c-ir}^{c+ir} \frac{1}{z\,\mathrm{Log}(z)}\,dz = 0$.