Aalto university
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Exercise sheet 10
Complex Analysis, MS-C1300.
Hand in exercise 1 and 2 in separate files for grading. Deadline Wednesday 22.11 at 23:59. The exercises should be uploaded to the correct folder on MyCourses as pdf-files with name and student number in the file name. **Submission via MyCourses is the only accepted way.** Done during class Thursday 23.11 or Friday 24.11.

(1) Let $(f_n)_{n=1}^{\infty}$ be the sequence of functions defined on $[0, 1]$ as follows:

$$f_n(t) = \begin{cases} 4n^2t, \text{ if } 0 \leq t \leq (1/2n) \\ 4n - 4n^2t, \text{ if } (1/2n) \leq t \leq (1/n) \\ 0, \text{ if } (1/n) \leq t \leq 1. \end{cases}$$

Draw a graph of $f_n(t)$, and check that

$$\lim_{n\to\infty} f_n(t) = 0,$$

but

$$\lim_{n\to\infty} \int_0^1 f_n(t)dt \neq 0.$$

(6p)

(2) Verify that

$$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^n}$$

converges normally in $\{z \in \mathbb{C}; |z| < 1\}$. Also verify that it diverges when $|z| > 1$. (It actually diverges when $|z| \geq 1$ but you don't need to worry about the points where $|z| = 1$.) (6p)

(3) Let D be a bounded domain in the complex plane. Suppose that every function in a sequence $(f_n)_{n=1}^{\infty}$ is continuous on $\overline{D}$ and analytic in D. Assume that this sequence converges uniformly on ∂D, and prove that it converges uniformly on D.

(4) Assume that $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ converges uniformly on a set A. Show that $(f_n + g_n)_{n=1}^{\infty}$ converges uniformly on A. Also show that $(f_n g_n)_{n=1}^{\infty}$ converges uniformly if we also assume that $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are uniformly bounded on A. (Here the sequences are uniformly bounded if there exists c such that $\sup(|f_n(z)|; z \in A) < c$ and $\sup(|g_n(z)|; z \in A) < c$ for all n.)