Aalto university
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Exercise sheet 9
Complex Analysis, MS-C1300.
Hand in exercise 1 and 2 in separate files for grading. Deadline Monday 20.11 at 23:59. The exercises should be uploaded to the correct folder on MyCourses as pdf-files with name and student number in the file name. **Submission via MyCourses is the only accepted way.** Done during class Tuesday 21.11 or Wednesday 22.11.

(1) Assume that a function f is analytic in an annulus D centered at a point z_0, that is
$$D = \{z \in \mathbb{C}\ ; a < |z - z_0| < b\},$$
where $0 \leq a < b \leq \infty$ and $z_0 \in \mathbb{C}$. Show that
$$\int_{|z-z_0|=r} f(z)\ dz = \int_{|z-z_0|=s} f(z)\ dz$$
whenever $a < r < s < b$ and the circles $|z - z_0| = r$ and $|z - z_0| = s$ are positively oriented. (6p)

(2) Let $\alpha(t) = e^{-it}$ and $\beta(t) = 3\cos t + i \sin t$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Compute
$$\int_{\beta+\alpha} \frac{16\mathrm{Log}(z)}{z(z-2)^2(z-4)}\ dz.$$
(6p)

(3) Let $\alpha(t) = e^{it}$, $\beta(t) = \frac{5}{3}+e^{it}$, and $\gamma(t) = -1+2e^{it}$ for $0 \leq t \leq 2\pi$.
   (a) Is the cycle $\sigma = (\alpha, \beta, \gamma)$ homologous to zero in $\mathbb{C}\setminus\{2i, -2i\}$?
   (b) Are the cycles $\sigma = (\alpha, -\gamma)$ and $\tau = (\beta, \beta)$ homologous in $\mathbb{C} \setminus \overline{\Delta(0, \frac{1}{2})}$?
   (c) Are the cycles $\sigma = (\alpha, \beta, \gamma)$ and $\tau = (\gamma, \gamma, \beta)$ homologous in $\mathbb{C} \setminus \{0\}$?

(4) Prove that
$$\lim_{r\to\infty} \int_{c-ir}^{c+ir} \frac{1}{z\,\mathrm{Log}(z)}\ dz = 0$$
when $c > 1$. (*Hint*: Consider $\int_\gamma (z\,\mathrm{Log}(z))^{-1}\ dz$ where $\gamma(t) = c + re^{it}$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.)