MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2023

WEEK 47: VIRTUAL WORK DENSITY

5 VIRTUAL WORK DENSITY

LEARNING OUTCOMES

Students are able to solve the lecture problems, home problems, and exercise problems on the topics of the week:

- ☐ The concepts, quantities, and equations of linear elasticity theory.
- ☐ Principle of virtual work for linear elasticity and virtual work densities.
- ☐ Virtual work densities of the solid, thin slab, bar, and torsion models.

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume.

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume.

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

5.1 LINEAR ELASTICITY

In the usual setting, a reference solution $(\overset{\leftrightarrow}{\sigma}^\circ, \vec{u}^\circ, V^\circ)$ with $\vec{u}^\circ = 0$ is assumed known. The goal is to find a new solution $(\overset{\leftrightarrow}{\sigma}, \vec{u}, V)$ corresponding to a slightly changed setting.

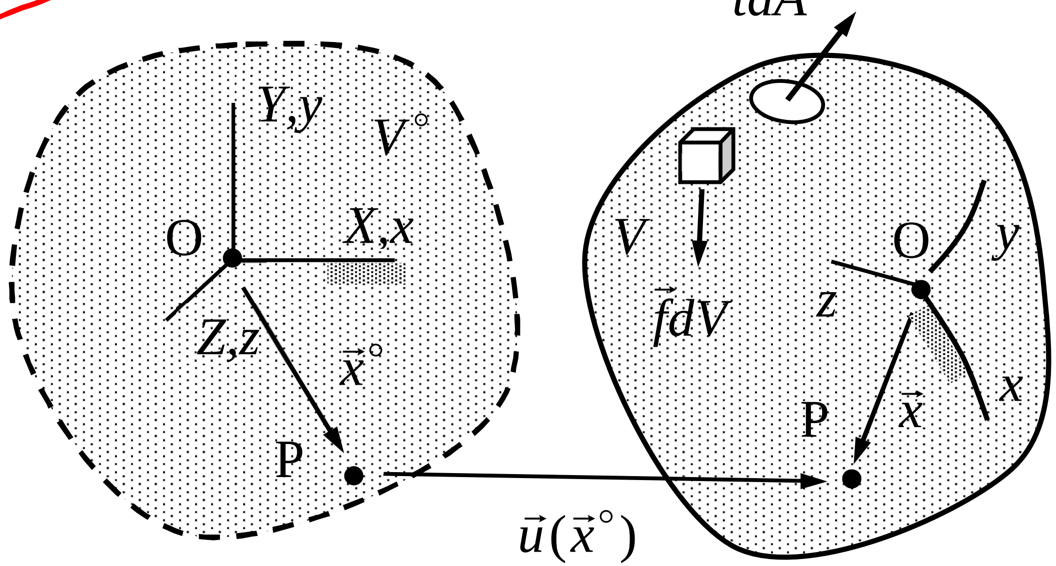
deformed body

$$\dot{m} = 0 \quad : \quad \rho^\circ = J\rho$$

$$\dot{\vec{p}} = \vec{F} \quad : \quad \nabla \cdot \overset{\leftrightarrow}{\sigma} + \vec{f} = 0 \quad \vec{r} \in V$$

$$\dot{\vec{p}} = \vec{F} \quad : \quad \vec{n} \cdot \overset{\leftrightarrow}{\sigma} = \vec{t} \quad \vec{r} \in \partial V_t$$

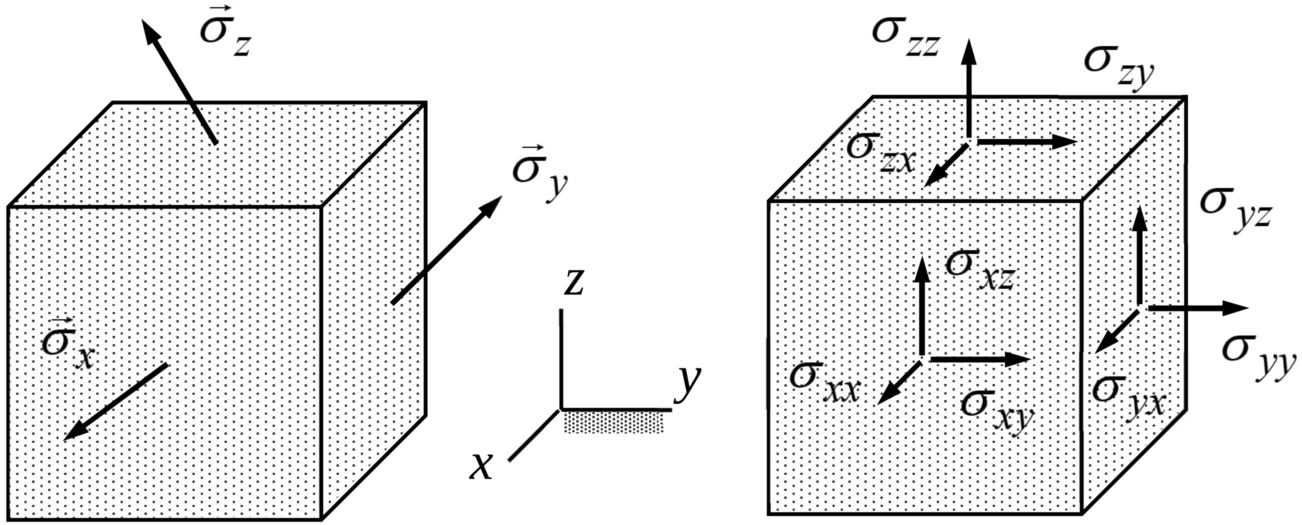
$$\dot{\vec{L}} = \vec{M} \quad : \quad \overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}_{\mathrm{c}} \quad \vec{r} \in V$$



A constitutive equation of type $g(\overset{\leftrightarrow}{\sigma}, \vec{u}) = 0$, bringing the material details into the model, and displacement boundary conditions are also needed.

TRACTION AND STRESS

Traction vector $\vec{\sigma} = \lim \Delta\vec{F} / \Delta A$ describes the internal force acting on a surface element in classical continuum mechanics. Stress tensor $\vec{\vec{\sigma}}$ describes all internal forces acting on a material element. The quantities are related by $\vec{\sigma} = \vec{n} \cdot \vec{\vec{\sigma}}$ where $\vec{n}$ is the unit normal to a material surface.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component (on opposite sides directions are the opposite).

GENERALIZED HOOKE'S LAW

The isotropic homogeneous material material model $g(\vec{\sigma},\vec{u})=0$ of the present course can be expressed, e.g., in its compliance form as

Strain-stress: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} = [E]^{-1}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$ and $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$

Strain-displacement: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}$ and $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$

Above, E is the Young's modulus, ν the Poisson's ratio, and $G = E/(2+2\nu)$ the shear modulus. Strain and stress are assumed to be symmetric.

MATERIAL PARAMETERS

Material	ρ **[** kg / m^3 **]**	E [GN / m^2]	ν **[** 1 **]**
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

EXAMPLE. Determine the *stress-strain* relationship of linear isotropic material subjected to (a) $xy-$plane stress and (b) $x-$axis stress (uni-axial) conditions. Start with the generic *strain-stress* relationships

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}.$$

Answer: (a) $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{Bmatrix}$ and $\sigma_{xy} = G\gamma_{xy}$ **(**b) $\sigma_{xx} = E\varepsilon_{xx}$

- In $xy-$plane stress, the stress components having at least one z as an index vanish (notice that the corresponding strain components need not to vanish)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{Bmatrix} \text{ and } \sigma_{xy} = G\gamma_{xy} \leftarrow$$

- In the $x-$axis stress, components having y or z as an index vanish:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ 0 \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \sigma_{xx} = E\varepsilon_{xx} \leftarrow$$

EXAMPLE Determine the *stress-strain* relationship of linearly elastic isotropic material subjected to (a) $xy-$plane stress and (b) $xy-$plane strain conditions. Start with the generic *strain-stress* relationship.

Answer: (a) $$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ where } [E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

(b) $$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_\varepsilon \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ with } [E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

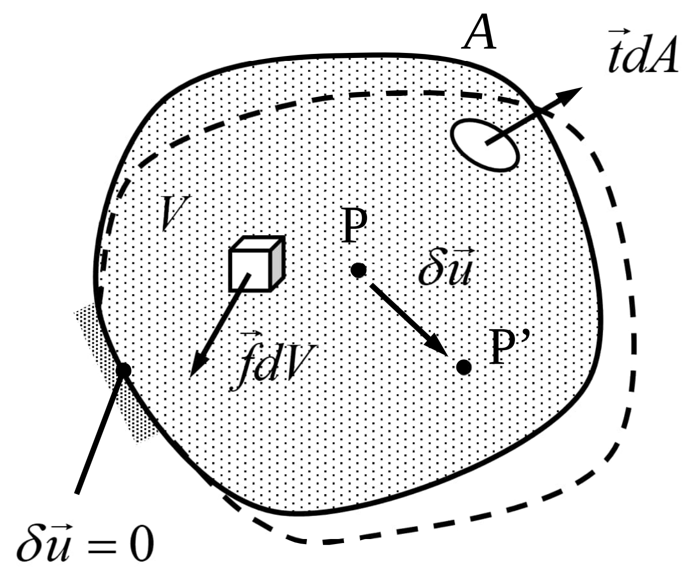
5.2 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\vec{u}$ is just one form of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\text{c}}) dV$$

$$\delta W_V^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV = \int_V (\vec{f} \cdot \delta\vec{u}) dV$$

$$\delta W_A^{\text{ext}} = \int_A \delta w_A^{\text{ext}} dA = \int_A (\vec{t} \cdot \delta\vec{u}) dA$$



The details of the final expressions may vary case by case, but the but the starting point is always the generic expressions above!

- Let us consider the balance law of momentum in its local form. Multiplication by the variation of the displacement, integration over the domain, and integration by parts give

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad \vec{r}\in V \quad \Leftrightarrow$$

$$\int_V (\nabla\cdot\vec{\vec{\sigma}}+\vec{f})\cdot\delta\vec{u}dV=\int_V (-\vec{\vec{\sigma}}:(\nabla\delta\vec{u})_{\mathrm{c}}+\vec{f}\cdot\delta\vec{u})dV+\int_A (\vec{n}\cdot\vec{\vec{\sigma}}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}$$

- Balance of momentum $\vec{t}=\vec{n}\cdot\vec{\vec{\sigma}}$ written for the boundary, local form of moment of momentum $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$, and the definition of linear strain $2\vec{\vec{\varepsilon}}=\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}$ give the final form

$$\delta W=-\int_V (\vec{\vec{\sigma}}:\delta\vec{\vec{\varepsilon}}_{\mathrm{c}})dV+\int_V (\vec{f}\cdot\delta\vec{u})dV+\int_A (\vec{t}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}\in U\,.$$

DEFINITIONS AND NOTATIONS

- **Domain** $\Omega \subset \mathbb{R}^n$**, boundary** $\partial\Omega$, and subset of the boundary $\partial\Omega_t$, $\partial\Omega_u$, etc.

- **Function sets:** $C^0(\Omega)$ (continuous functions on Ω), $C^1(\Omega)$, $L_2(\Omega)$ etc.

- **Notations** $\exists$ ~ "exists" & $\forall$ ~ "for all" & $\vee$ ~ "or" & $\wedge$ ~ "and"

- **Fundamental theorem of calculus** (integration by parts) $u, v \in C^0(\Omega)$

$$\int_\Omega u \frac{\partial v}{\partial \alpha} d\Omega = \int_{\partial\Omega} (n_\alpha u v) d\Gamma - \int_\Omega v \frac{\partial u}{\partial \alpha} d\Omega \quad \alpha \in \{x, y, z, \ldots\}$$

- **Fundamental lemma of variation calculus** $u, v \in C^0(\Omega)$

$$\int_\Omega uv d\Omega = 0 \;\; \forall v \;\; \Leftrightarrow \;\; u = 0 \text{ in } \Omega$$

VIRTUAL WORK DENSITIES

Virtual work densities of the internal forces, external volume forces, and external surface forces are (subscripts V and A denote virtual work per unit volume and area, respectively)

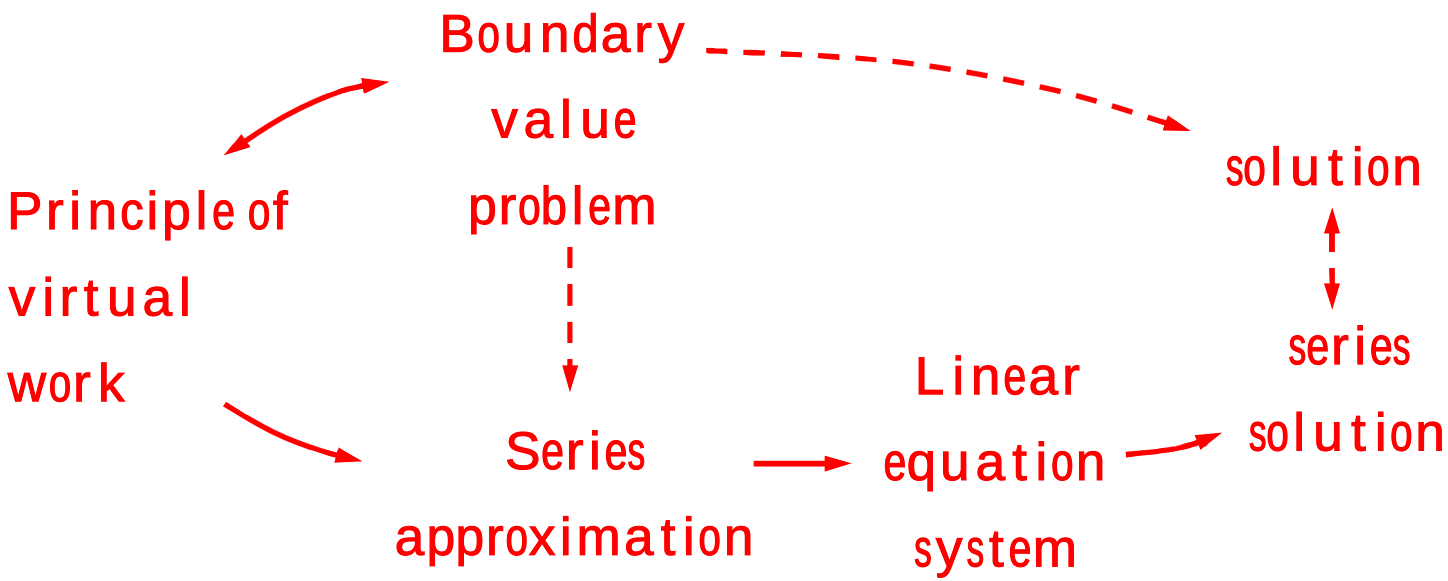
Internal forces: $\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$

External forces: $\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$

Virtual work densities consist of terms containing kinematic quantities and their “work conjugates”!

PRINCIPLE OF VIRTUAL WORK IN MECHANICS

The common dimension reduction (engineering) models like beam, plate, shell etc. models have their origin in the principle of virtual work priciple. The principle is also the starting point for numerical solution methods of various types: A series approximation is substituted there to end up with an algebraic equations system for the unknown parameters.



5.3 ENGINEERING MODELS

Engineering (dimension reduction) models are defined concisely by their *kinematic* and *kinetic* assumptions. The rest is pure mathematics based on the principle of virtual work.

Bar: $\vec{u}(x,y,z)=\vec{u}_0(x)$ $\qquad$ $\sigma_{xx}\neq 0$ only

Beam: $\vec{u}(x,y,z)=\vec{u}_0(x)+\vec{\theta}_0(x)\times\vec{\rho}(y,z)$ $\qquad$ $\sigma_{yy}=\sigma_{zz}=0$

Curved beam: $\vec{u}(s,n,b)=\vec{u}_0(s)+\vec{\theta}_0(s)\times\vec{\rho}(n,b)$ $\qquad$ $\sigma_{nn}=\sigma_{bb}=0$

Thin slab: $\vec{u}(x,y,z)=\vec{u}_0(x,y)$ $\qquad$ $\sigma_{zz}=\sigma_{yz}=\sigma_{zx}=0$

Membrane: $\vec{u}(s,n,b)=\vec{u}(s,n)$, $\qquad$ $\sigma_{ss}\neq 0$, $\sigma_{nn}\neq 0$, $\sigma_{sn}\neq 0$ only

Plate: $\vec{u}(x,y,z)=\vec{u}(x,y)+\vec{\theta}(x,y)\times\vec{\rho}(z)$ $\qquad$ $\sigma_{zz}=0$

Shell: $\vec{u}(s,n,b)=\vec{u}(s,n)+\vec{\theta}(s,n)\times\vec{\rho}(b)$ $\qquad$ $\sigma_{bb}=0$

VIRTUAL WORK DENSITY OF A MODEL

Virtual work density δw serves as a concise representation of the model in the recipe for the element contribution. To derive the virtual work density (bar, beam plate, shell etc.)

- ☐ Start with the virtual work expression $\delta W = \int_V \delta w_V dV$ of an elastic body. Use the kinematical and kinetic assumptions of the model to simplify δw_V. After that,

- ☐ integrate over the small dimension(s) to end up with expression $\delta W = \int_\Omega \delta w_\Omega d\Omega$, where the remaining integral is over the mathematical solution domain Ω. Then, virtual work density of the model is δw_Ω.

The dimension of domain Ω is smaller than that of the physical domain due to the integration over the small dimensions.

5.4 SOLID MODEL

Solid model does not contain any assumptions in addition to those of the generic linear elasticity theory. Therefore

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} [E]\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix},$$

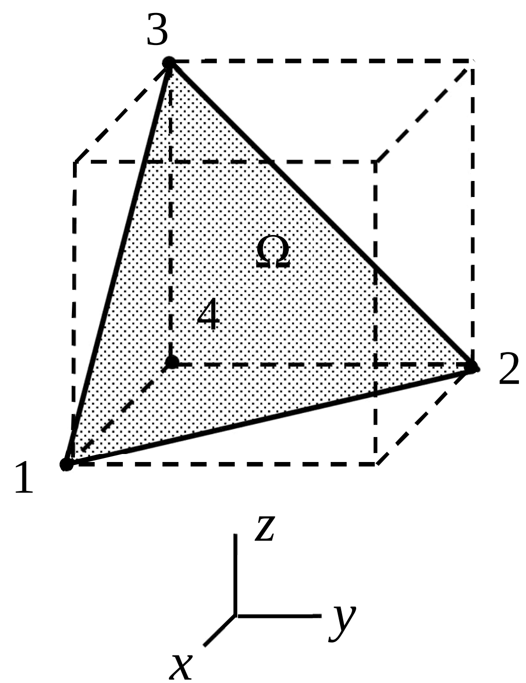
$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

The simplest element is a four-node tetrahedron with linear interpolations to the three displacement components $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$.

- Usually, the approximations to the components are of the same type. For example, the linear element interpolant for a tetrahedron element shape is given by

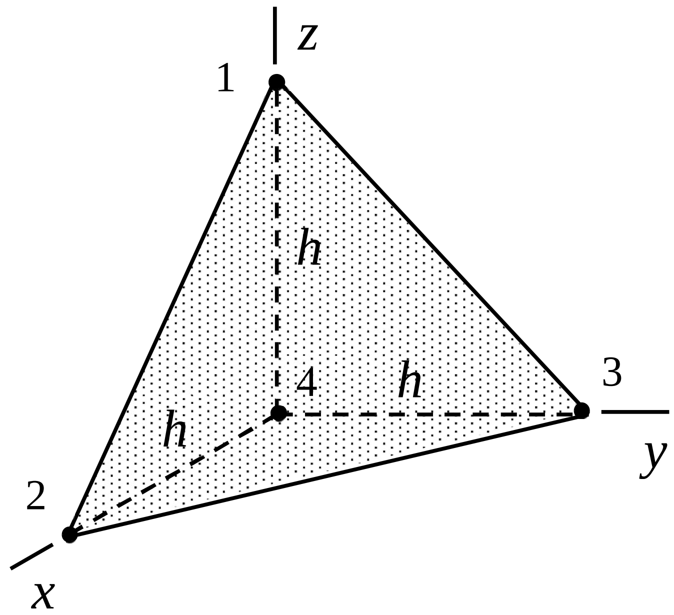
$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \text{ where}$$

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}.$$



The solid model works with any geometry but use of plate, shell, beam, bar etc. models may mean huge savings in computational complexity as dimension of the mathematical solution domain is smaller than 3 (physical dimensions)!

EXAMPLE 5.1 Compute the virtual work expression of external volume force with the components $f_x = \text{constant}$ and $f_y = f_z = 0$. Consider the tetrahedron element shown and assume that the shape functions are linear (a four-node tetrahedron element).



Answer: $\delta W_V^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \dfrac{f_x h^3}{24} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$

- The linear shape functions can be deduced directly from the figure $N_1 = z/h$, $N_2 = x/h$, $N_3 = y/h$ and $N_4 = 1 - x/h - y/h - z/h$ (sum of the shape functions is 1). Therefore, the approximation is given by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} N_1 u_{x1} + N_2 u_{x2} + N_3 u_{x3} + N_4 u_{x4} \\ N_1 u_{y1} + N_2 u_{y2} + N_3 u_{y3} + N_4 u_{y4} \\ N_1 u_{z1} + N_2 u_{z2} + N_3 u_{z3} + N_4 u_{z4} \end{Bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}.$$

- Virtual work density of the external volume forces is

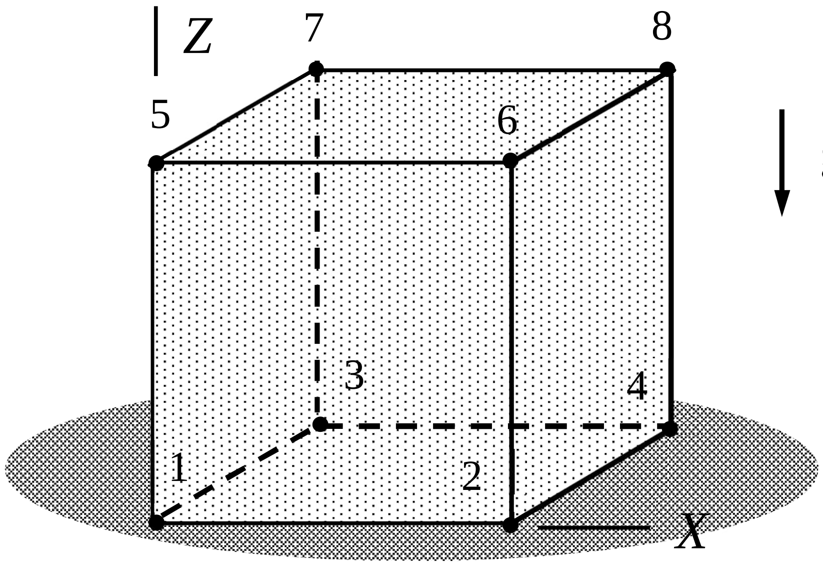
$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}^{\text{T}} \begin{bmatrix} \delta u_{x1} & \delta u_{y1} & \delta u_{z1} \\ \delta u_{x2} & \delta u_{y2} & \delta u_{z2} \\ \delta u_{x3} & \delta u_{y3} & \delta u_{z3} \\ \delta u_{x4} & \delta u_{y4} & \delta u_{z4} \end{bmatrix} \begin{Bmatrix} f_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} f_x$$

- Virtual work expression of the external volume force is obtained as an integral over the volume:

$$\delta W^{\mathrm{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \int_V \begin{Bmatrix} z/h \\ x/h \\ y/h \\ 1-x/h-y/h-z/h \end{Bmatrix} f_x dV = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h^3}{24} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}.$$

The explicit form of the virtual work expression for a generic shape is too complicated to be practical (due to the large number of the geometric parameters involved).

EXAMPLE 5.2 A concrete cube of edge length L, density ρ, and elastic properties E, ν is subjected to its own weight on a horizontal floor. Calculate the displacement of the top surface with one hexahedron element and tri-linear approximation. Assume that displacement components in $X-$ and $Y-$directions vanish, $u_{Z5} = u_{Z6} = u_{Z7} = u_{Z8}$, and that the bottom surface is fixed.



Answer: $u_{Z5} = -\frac{1}{2}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}$

- Let the material coordinate system coincide with the structural system. The shape functions for the upper surface nodes can be deduced directly from the figure. Approximations to the displacement components are $(\xi = x/L,\ \eta = y/L,\ \zeta = z/L)$

$$u = 0,\ v = 0,\ \text{and}\ \ w = \begin{Bmatrix} (1-\xi)(1-\eta)\zeta \\ \xi(1-\eta)\zeta \\ (1-\xi)\eta\zeta \\ \xi\eta\zeta \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{Z5} \\ u_{Z5} \\ u_{Z5} \\ u_{Z5} \end{Bmatrix} = \frac{z}{L} u_{Z5},\ \text{giving}\ \frac{\partial w}{\partial z} = \frac{1}{L} u_{Z5}\ .$$

- When the approximation is substituted there, the virtual work densities of the internal forces, external forces, and their sum simplify to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \partial \delta w / \partial z \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \partial w / \partial z \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ z/L\delta u_{Z5} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix},$$

$$\delta w_V = \delta w_V^{\text{int}} + \delta w_V^{\text{ext}} = -\delta u_{Z5}[\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{u_{Z5}}{L^2} + \frac{z}{L}\rho g].$$

- Virtual work expression is obtained as integral of the density over the volume:

$$\delta W = \int_V \delta w_V dV = -\delta u_{Z5}[\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} L u_{Z5} + \frac{L^3}{2}\rho g].$$

- Finally, principle of virtual work $\delta W = 0 \ \forall \delta u_{Z5}$ implies that

$$u_{Z5} = -\frac{1}{2}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}.$$

5.5 THIN SLAB MODEL

Thin slab model is the in-plane mode of the plate model and also the solid model in two dimensions. The elasticity matrices $[E]_\sigma$ and $[E]_\varepsilon$ for the plane stress and plane strain versions differ.

Internal forces: $$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix},$$

External forces: $$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}.$$

Although the virtual work density $\delta w_{\partial\Omega}^{\text{ext}}$ for the external line force t_x, t_y acting on the edges belongs to the thin slab, it will be treated separately (like point forces/moments)!

- Thin slab is a body which is thin in one dimension. The kinematic assumptions of the thin slab model $u_x = u(x,y)$ and $u_y = v(x,y)$ give the (non-zero) strain components

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}.$$

- The kinetic assumptions of the plane stress version $\sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0$ (these are replaced by kinematic assumptions $\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$ in the plane strain version) and the generalized Hooke's law imply the stress-strain relationship

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \equiv [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}.$$

- Therefore, the generic virtual work densities (per unit volume) simplify first to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \quad \text{and} \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}.$$

- Integration over the volume is performed in two steps: first over the thickness with $z \in [z_-, z_+]$ $(t = |z_+ - z_-|)$ and after that over the mid-plane $(x, y) \in \Omega$. As the virtual work density of internal forces does not depend on z and $dV = dzd\Omega$

$$\delta W^{\text{int}} = \int_\Omega (\int_{z_-}^{z_+} \delta w_V^{\text{int}} dz) d\Omega = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}. \quad \Leftarrow$$

- The contributions for the external forces follow in the same manner. Boundary of the body is divided into the lower, upper and edge parts A_-, A_+, $S = \partial\Omega \times [z_-, z_+]$, Surface area elements are $dA = d\Omega$ (upper and lower surfaces) and $dA = dzds$ (edge). The volume force acting on V and the surface forces on A_- and A_+ give

$$\delta W^{\text{ext}} = \int_{\Omega} \left(\int_{z_-}^{z_+} \delta w_V^{\text{ext}} dz + \sum_{z \in \{z_-, z_+\}} \delta w_A^{\text{ext}}\right) d\Omega = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \text{in which}$$

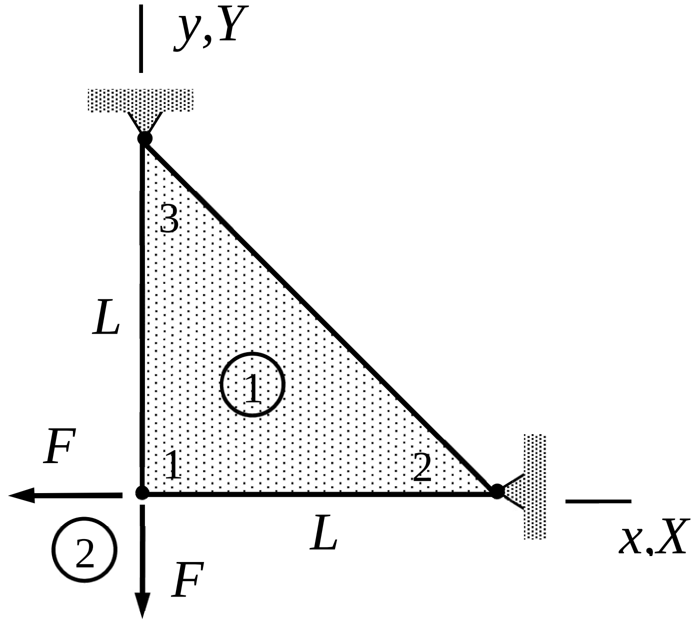
$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \left(\int_{z_-}^{z_+} \delta w_V^{\text{ext}} dz \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} + \sum_{z \in \{z_-, z_+\}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}\right) = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}. \qquad \Leftarrow$$

- The contribution from the remaining boundary part $S = \partial\Omega \times [z_-, z_+]$

$$\delta W^{\text{ext}} = \int_{\partial\Omega} \int_{z_-}^{z_+} \delta w_A^{\text{ext}} dz ds = \int_{\partial\Omega} \delta w_{\partial\Omega}^{\text{ext}} ds \quad \text{where} \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} \qquad \Leftarrow$$

is part of the virtual work expression for the thin slab model. In practice, the distributed boundary force is taken into account by force elements by using the restriction of the element approximation to the boundary. With a linear or bilinear element, distributed force an element edge gives rise to a two-node force element.

EXAMPLE 5.3 Derive the virtual work expression for the linear triangle element shown. Young's modulus E, Poisson's ratio ν, and thickness t are constants. Distributed external force vanishes. Assume plane strain conditions. Also determine the displacement of node 1 when the force components acting on the node are as shown in the figure.



Answer: $$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{Et}\frac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

- Nodes 2 are 3 are fixed so the non-zero displacement components are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L-x-y}{L}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \text{so} \quad \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{Bmatrix} = -\frac{1}{L}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u/\partial y \\ \partial v/\partial y \end{Bmatrix} = -\frac{1}{L}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work density of internal forces is given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{tE}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1}+u_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2} \frac{tE}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix} \Leftrightarrow$$

$$\delta W^1 = -\frac{Et}{4(1+\nu)(1-2\nu)} \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 3-4\nu & 1 \\ 1 & 3-4\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}, \quad \leftarrow$$

$$\delta W^2 = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ -F \end{Bmatrix}. \quad \leftarrow$$

- Principle of virtual work in the form $\delta W = \delta W^1 + \delta W^2 = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

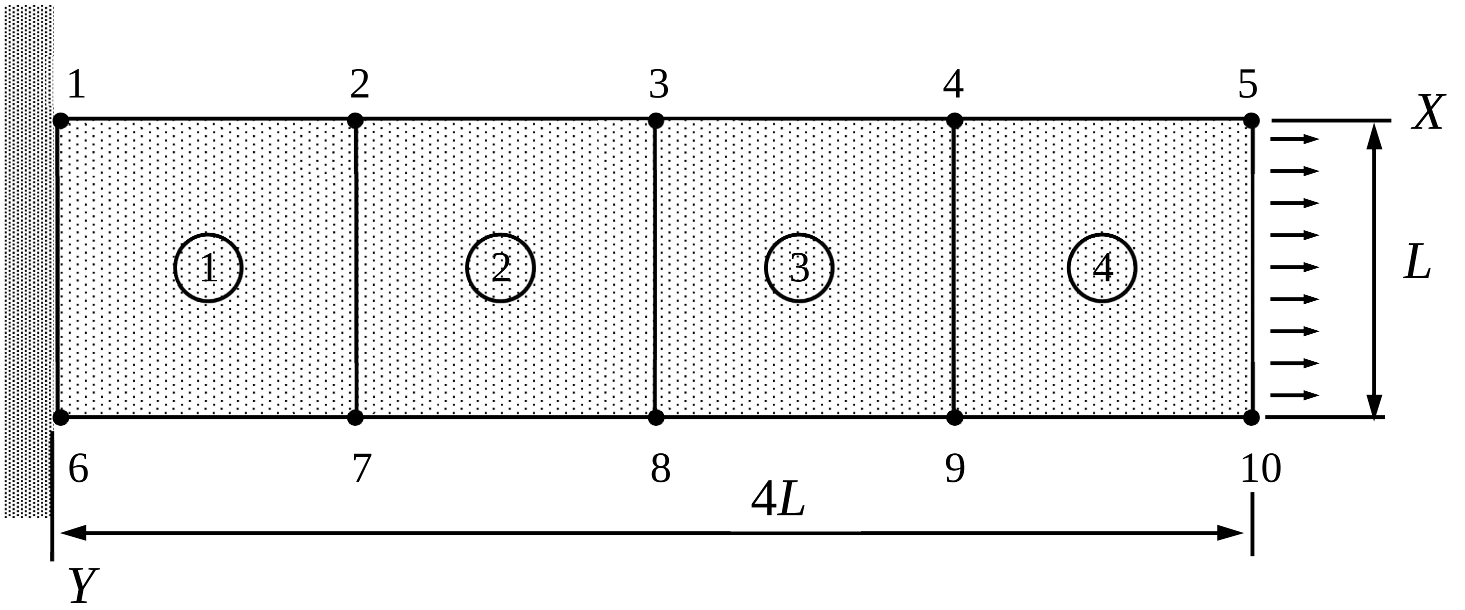
$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{Et}{4(1+\nu)(1-2\nu)} \begin{bmatrix} 3-4\nu & 1 \\ 1 & 3-4\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right) = 0 \quad \forall \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix} \Rightarrow$$

$$\frac{Et}{4(1+\nu)(1-2\nu)}\begin{bmatrix} 3-4\nu & 1 \\ 1 & 3-4\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{Et}\frac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

NOTICE: The point forces acting on a thin slab should be considered as "equivalent nodal forces" i.e. just representations of distributed forces acting on the edges with some selection of the element division. Therefore, refinement of the mesh requires a new set of equivalent nodal forces. Under the action of an actual point force, exact solution to the displacement becomes non-bounded so also the numerical solution to the displacement at the point of action increases without a bound, when the mesh is refined.

EXAMPLE 5.4 A thin slab is loaded by an evenly distributed traction having the resultant F as shown. Calculate the displacement at the midpoint of edge 5-10 by using bi-linear approximation in each element and the plane-stress assumption. Thickness of the slab is t. Material parameters E and $\nu = 1/3$ are constants.



Answer: $\dfrac{u_{X5} + u_{X10}}{2} = \dfrac{6856}{1731}\dfrac{F}{Et} \approx 3.96072\dfrac{F}{Et}$ (bar model or $\nu = 0$ gives $4\dfrac{F}{Et}$)

- Mathematica solution can be obtained with the problem description tables

	model	properties	geometry
1	PLANE	{{E, ν}, {t}}	Polygon[{1, 2, 7, 6}]
2	PLANE	{{E, ν}, {t}}	Polygon[{2, 3, 8, 7}]
3	PLANE	{{E, ν}, {t}}	Polygon[{3, 4, 9, 8}]
4	PLANE	{{E, ν}, {t}}	Polygon[{4, 5, 10, 9}]
5	FORCE	$\{\{\frac{F}{L}, \frac{F}{L}\}, \{0, 0\}, \{0, 0\}\}$	Line[{5, 10}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], uY[2], 0}	{0, 0, 0}
3	{2 L, 0, 0}	{uX[3], uY[3], 0}	{0, 0, 0}
4	{3 L, 0, 0}	{uX[4], uY[4], 0}	{0, 0, 0}
5	{4 L, 0, 0}	{uX[5], uY[5], 0}	{0, 0, 0}
6	{0, L, 0}	{0, 0, 0}	{0, 0, 0}
7	{L, L, 0}	{uX[7], uY[7], 0}	{0, 0, 0}
8	{2 L, L, 0}	{uX[8], uY[8], 0}	{0, 0, 0}
9	{3 L, L, 0}	{uX[9], uY[9], 0}	{0, 0, 0}
10	{4 L, L, 0}	{uX[10], uY[10], 0}	{0, 0, 0}

5.6 BAR MODEL

Bar model is one of the loading modes of the beam model and the solid model in one dimension. Virtual work densities of the model are given by

Internal forces: $\delta w_{\Omega}^{\text{int}} = -\dfrac{d\delta u}{dx} EA \dfrac{du}{dx}$

External forces: $\delta w_{\Omega}^{\text{ext}} = \delta u f_x, \quad \delta w_{\partial\Omega}^{\text{ext}} = \delta u F_x.$

Although the virtual work density $\delta w_{\partial\Omega}^{\text{ext}}$ for the external point force F_x acting on the edges (here on the nodes) belongs to the bar model, it will be treated separately by forces/moments elements.

- A bar is a thin body in two dimensions. The kinematic and kinetic assumptions of the bar model $\vec{u}(x,y,z)=\vec{u}(x)$ and only $\sigma_{xx}\neq 0$ imply the non-zero strain and stress components

$$\varepsilon_{xx}=\frac{du}{dx} \text{ and } \sigma_{xx}=E\varepsilon_{xx}=E\frac{du}{dx}.$$

- Therefore, virtual work densities of the solid model simplify to

$$\delta w_V^{\text{int}}=-\begin{Bmatrix}\delta\varepsilon_{xx}\\ \delta\varepsilon_{yy}\\ \delta\varepsilon_{zz}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix}-\begin{Bmatrix}\delta\gamma_{xy}\\ \delta\gamma_{yz}\\ \delta\gamma_{zx}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix}=-\frac{d\delta u}{dx}E\frac{du}{dx},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = f_x \delta u \quad \text{and} \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = t_x \delta u .$$

- Integration over the body consists of integration over the cross-section (small dimensions of the bar and beam models) and integration over the length

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = \int_\Omega (\int_A \delta w_V^{\text{int}} dA) d\Omega = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} . \quad \Leftarrow$$

- The contributions of the external forces are obtained in the same manner. Considering first the volume force and the surface force acting on the circumferential part (in the final form, f_x denotes force per unit length although the symbol is the same as for the volume force)

$$\delta W^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV + \int_A \delta w_A^{\text{ext}} dA = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{ext}} = \delta u \left(\int_A f_x dA + \int_S t_x ds \right) = \delta u f_x .$$

- Surface forces on the remaining area (end surfaces) give (in the final form, F_x is force acting at an end point in the direction of the axis)

$$\delta W^{\text{ext}} = \sum_{\partial\Omega} \int_A \delta w_A^{\text{ext}} dA = \sum_{\partial\Omega} \delta u [\int_A t_x dA] = \sum_{\partial\Omega} \delta u F_x .$$

Virtual work of traction at the end surfaces belongs to the bar model but the contribution is taken into account by a force element of one node.

5.7 TORSION MODEL

Torsion model is one of the loading modes of beam. Virtual work densities of the model are given by

Internal forces: $\delta w_{\Omega}^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx}$

External forces: $\delta w_{\Omega}^{\text{ext}} = \delta\phi m_x, \;\; \delta w_{\partial\Omega}^{\text{ext}} = \delta\phi M_x$

Although the virtual work density $\delta w_{\partial\Omega}^{\text{ext}}$ for the external point moment M_x acting on the edges (here on the nodes) belongs to the torsion model, it will be treated separately by moment elements.

- The kinematic assumptions of the torsion model $u_x = 0$, $u_y = -z\phi(x)$ and $u_z = y\phi(x)$ follow from the kinematic assumption of the beam model when only $\phi(x) \neq 0$. The strain-displacement relationships and the generalized Hooke's law give

$$\gamma_{xy} = -z\frac{d\phi}{dx},\ \ \gamma_{zx} = y\frac{d\phi}{dx},\ \ \sigma_{xy} = G\gamma_{xy} = -Gz\frac{d\phi}{dx},\ \text{and}\ \sigma_{zx} = G\gamma_{zx} = Gy\frac{d\phi}{dx}.$$

- Virtual work densities of internal and external forces follow from the generic expressions

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = -\frac{d\delta\phi}{dx} G(z^2 + y^2)\frac{d\phi}{dx}$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = \delta\phi(yf_z - zf_y) \quad \text{and} \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = \delta\phi(yt_z - zt_y).$$

- Virtual work expressions are integrals over the volume divided here as integrals over the cross-section and length. Assuming that the shear modulus is constant

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = \int_\Omega (\int_A \delta w_V^{\text{int}} dA) d\Omega = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} \int_A (z^2 + y^2) G dA \frac{d\phi}{dx} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx}. \quad \Leftarrow$$

The geometrical quantity $J = \int_A (z^2 + y^2) dA$ is called as the polar moment of the cross-section.

- The contribution from the external forces are obtained in the same manner (volume forces in V and surface forces on the entire A have to be accounted for). The volume and the circumferential area give

$$\delta W^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV + \int_A \delta w_A^{\text{ext}} dA = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{ext}} = \delta\phi[\int_A (yf_z - zf_y) dA + \int_S (yt_z - zt_y) ds] = \delta\phi m_x \ . \ \Leftarrow$$

In the final form above, m_x is the moment per unit length.

- Surface forces on the remaining area (end surfaces) give rise to external moments M_x acting at the ends. These point moments are treated by using one-node force-moment elements.