Problem 11.1: Interior Point Method for Linear Optimization

Consider the following (primal) Linear Optimization problem in standard form

$$(P): \quad \min_{x}. \; c^\top x \tag{1}$$
$$\text{subject to: } Ax = b \tag{2}$$
$$x \geq 0 \tag{3}$$

with variables $x \in \mathbb{R}^n$ and problem data $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. The dual of the problem (1) – (3) can be written as

$$(D): \quad \max_{v,u}. \; b^\top v \tag{4}$$
$$\text{subject to: } A^\top v + u = c \tag{5}$$
$$u \geq 0 \tag{6}$$

with the dual variables $v \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$. In this exercise, we want to solve the problem (1) – (3) using a primal-dual path following Interior Point Method (IPM). An optimal solution to both primal and dual problems satisfies the KKT conditions of P:

$$\begin{aligned} Ax &= b, & x &\geq 0, & &\text{(primal feasibility)} \\ A^\top v + u &= c, & u &\geq 0, & &\text{(dual feasibility)} \\ u^\top x &= 0. & & & &\text{(complementarity)} \end{aligned}$$

To solve (1) – (3) with the IPM, we will use a logarithmic barrier function. Adding this barrier function to the problem P, it becomes

$$(BP): \quad \min. \; c^\top x - \mu \sum_{i=1}^{n} \log(x_i) \tag{7}$$
$$\text{subject to: } Ax = b, \tag{8}$$

where $\mu > 0$ is a suitable penalty parameter. The basic idea of IPMs is to initially solve BP with a large value of μ, iteratively reduce its value, and re-solve it until we are close enough to the optimum. We will solve (7) – (8) with a primal-dual path following variant of IPM in which we solve a single Newton step for each value of μ.

(a) Write the KKT conditions of the problem (7) – (8) and motivate a suitable stopping criterion for this IPM variant.

(b) Let $w = [x, v, u]^\top$, and let $H(\overline{w}) = 0$ denote the KKT system of part (a) for any solution $\overline{w} = [\overline{x}, \overline{v}, \overline{u}]^\top$ and penalty parameter μ. Let $J(\overline{w})$ be the Jacobian of $H(w)$ at $\overline{w}$, and let $d_w = (w - \overline{w}) = [d_x, d_v, d_u]^\top$ be the direction vector. Using this notation, apply Newton-Raphson method to solve $H(w) = 0$ at $\overline{w}$ and derive formulas for d_v, d_u, and d_x.

(c) At iteration k, we will solve $d_{w^{k+1}} = [d_{x^k}, d_{v^k}, d_{u^k}]^\top$ based on the update formulas derived in part (b) (i.e., Newton step), update the solution $w^{k+1} = w^k + d_{w^{k+1}}$, and set $\mu^{k+1} = \beta\mu^k$ for some $\beta \in (0, 1)$ until the stopping criterion derived in part (a) is satisfied.

To start the IPM, we need an initial primal solution x^0 (which does not necessarily need to be feasible). Propose a method to find a strictly feasible primal solution x^0 for (1) – (3).

(d) Solve the problem starting form the initial solution x^0 computed in part (c). Implementation of this IPM variant can be downloaded here.