Exercise sheet 11

1) Determine the disks of convergence of:

a) $\sum_{n=1}^{\infty} \sqrt[n]{n}\,(z-1)^n$

b) $\sum_{n=1}^{\infty} n^2 (z+2)^{2^n}$

c) $\sum_{n=1}^{\infty} n!\,(z-i)^{n!}$

Solution: a) The disk of convergence is $\Delta(1,\rho)$ where $\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ for $a_n = \sqrt[n]{n}$.

We try to calculate $\lim_{n\to\infty} \sqrt[n]{a_n}$

$$\sqrt[n]{a_n} = (a_n)^{1/n} = \left(e^{\frac{1}{n}\ln n}\right)^{1/n} = e^{\frac{1}{n^2}\ln n}$$

Hence $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\frac{1}{n^2}\ln n} = e^0 = 1$

and the disk of convergence is $\Delta(1,1) =$

$= \{ z\in\mathbb{C};\ |z-1|<1\}$

b) The disk of convergence is $\Delta(-2,\rho)$ where $\rho^{-1} = \limsup_{n\to\infty} \sqrt[n]{a_n}$. Here $a_n = \begin{cases} k^2 & \text{if } n=2^k \\ 0 & \text{if } n\neq 2^k \end{cases}$

We calculate $\lim_{k\to\infty} \sqrt[2^k]{k^2} = \lim_{k\to\infty} \left(e^{2\ln k}\right)^{\frac{1}{2^k}} =$

$$= \lim_{k\to\infty} e^{\frac{\ln k}{2^{k-1}}} = e^0 = 1$$

$\limsup_{n\to\infty} \sqrt[n]{a_n}$ is not changed if we drop 0's from the sequence. Hence $\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{k\to\infty} \sqrt[2^k]{k^2} = 1$

and $\rho = 1$. That is,

$\Delta(-2,1)$ is the disk of convergence.

c) The disk of convergence is $\Delta(i, \rho)$ where $\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{a_n}$ where $a_n = \begin{cases} k & \text{if } k = n! \\ 0 & \text{if } k \neq n! \end{cases}$. We calculate

$$\lim_{k\to\infty} \sqrt[k]{k} = \lim_{k\to\infty} e^{\frac{1}{k}\ln k} = e^0 = 1.$$

Again, since dropping 0's doesn't effect $\limsup_{n\to\infty} \sqrt[n]{a_n}$ we get $\rho^{-1} = 1$ so the disk of convergence is $\Delta(i, 1)$.

(2) Find the Taylor series expansion of f about the origin, and identify the largest open disk $\Delta(0, \rho)$ in which the expansion is valid:

a) $f(z) = \dfrac{1}{(1-z)^2}$

b) $f(z) = \dfrac{1}{(1+z^2)^3}$

c) $f(z) = \mathrm{Log}(1+z^2)$

(Hint: Remember the geometric serie. Also remember that Taylor series can be differentiated and integrated termwise)

Solution: a) We know that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ in $|z|<1$. Since $\frac{d}{dz}(1-z)^{-1} = (1-z)^{-2}$

we get $\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{k=0}^{\infty} (k+1) z^k$

valid in $|z|<1$.

b) We keep differentiating and get

$$\frac{d}{dz}(1-z)^{-2} = +2(1-z)^{-3} = \frac{2}{(1-z)^3}.$$

That is, $\frac{1}{(1-z)^3} = \frac{1}{2}\sum_{k=1}^{\infty}(k+1)k\, z^{k-1} = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2} z^n$

Hence $\frac{1}{(1+z^2)^3} = \frac{1}{(1-(-z^2))^3} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} z^{2n}$

valid when $|z|<1$.

c) If we integrate $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ we get

$-\operatorname{Log}(1-z) = c + \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$. Since $\operatorname{Log}(1) = 0$ we have

$$\operatorname{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

when $|z|<1$.

Now, $\operatorname{Log}(1+z^2) = \operatorname{Log}(1-(-z^2)) =$

$$= -\sum_{n=1}^{\infty} \frac{(-z^2)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{2n} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{2n} \quad \text{when } |z|<1.$$

(3) Show that $\dfrac{e^z}{1-z} = \displaystyle\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!}\right) z^n$ when $|z|<1$. (Hint: Taylor series can be multiplied in their disks of convergence.)

Solution:

If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ then

$$f(z)g(z) = \sum c_n z^n \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Since $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ and $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$

we see that $c_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!} \cdot 1 = \sum_{k=0}^{n} \frac{1}{k!}$

That is, $\dfrac{e^z}{1-z} = \displaystyle\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!}\right) z^n$ when $|z|<1$

(since $\dfrac{1}{1-z} = \sum z^n$ in $|z|<1$ only)

(4) Assume that f is a non-constant entire function. Assume there is a constant $\lambda \neq 1$ such that $f(\lambda z) = f(z)$ for every $z \in \mathbb{C}$. Show that there must be an integer $m \geq 2$ such that $\lambda^m = 1$. Also show that there is an entire function g such that $f(z) = g(z^m)$ where m is the minimal integer $m \geq 2$ such that $\lambda^m = 1$.

Solution: $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{C}$ since f is entire. Since f is non-constant at least one coefficient $a_n \neq 0$, $n \geq 1$. Choose the coefficient with minimal index m. Since $f(\lambda z) = f(z)$ and the Taylor expansion is unique we get

$$a_m = \lambda^m a_m \Rightarrow \lambda^m = 1.$$

We also see that $a_k = 0$ unless $\lambda^k = 1$ and $k \geq 0$. Also $a_1 = 0$ since $a_1 = \lambda a_1$ and $\lambda \neq 1$. Choose m as the minimal k so that $k > 1$ and $\lambda^k = 1$.

We see that, since m is minimal, that $a_k = 0$ unless $k = lm$ for some $l \in \mathbb{N}$. Therefore $f(z) = \sum_{l=0}^{\infty} a_{lm} z^{lm}$.

Define $b_n = a_{nm}$ and put $g(z) = \sum_{n=0}^{\infty} b_n z^n$. We get $g(z^m) = \sum_{n=0}^{\infty} b_n z^{nm} = f(z)$.