Exercise sheet 12

(1) Obtain the Laurent series expansions of $f(z) = \frac{1}{2z - z^2}$ and $g(z) = \frac{2-2z}{(2z-z^2)^2}$ in

a) $\Delta^*(0,2) = \{ z \in \mathbb{C} ; 0 < |z| < 2 \}$

b) $\Delta^*(2,2) = \{ z \in \mathbb{C} ; 0 < |z-2| < 2 \}$

c) $D = \{ z \in \mathbb{C} ; |z| > 2 \}$

Solution: Notice that $f'(z) = \frac{d}{dz} (2z - z^2)^{-1} =$

$$= \frac{2-2z}{(2z-z^2)^2} = g(z)$$

We need to find Laurent series for f and then differentiate to find also Laurent series for g.

a) $$f(z) = \frac{1}{2z} \cdot \frac{1}{1 - \frac{z}{2}} \underset{|z|<2}{=} \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \frac{1}{2z} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+2}}$$

Therefore

$$f'(z) = g(z) = -\frac{1}{2z^2} + \sum_{n=1}^{\infty} \frac{n z^{n-1}}{2^{n+2}} = -\frac{1}{2z^2} + \sum_{n=0}^{\infty} \frac{(n+1) z^n}{2^{n+3}}$$

Both series are valid in $\Delta^*(0,2)$.

b) $\Delta^*(2,2) = \{ z ;\ 0 < |z-2| < 2 \}$

$$f(z) = -\frac{1}{z-2} \cdot \frac{1}{z} = \frac{-1}{z-2} \cdot \frac{1}{2+(z-2)} = -\frac{1}{2(z-2)} \cdot \frac{1}{1+\frac{(z-2)}{2}}$$

$$\underset{|z-2|<2}{=} -\frac{1}{2(z-2)} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (z-2)^n =$$

$$= \frac{-1/2}{z-2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} (z-2)^n \quad \text{in } \Delta^*(2,2)$$

$$f'(z) = g(z) = \frac{1/2}{(z-2)^2} + \sum_{n=0}^{\infty} \frac{(n+1)(-1)^n}{2^{n+2}} (z-2)^n$$

c) $D = \{ z ;\ |z| > 2 \}$. We look for a series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{valid in } D.$$

$$f(z) = \frac{1}{2z - z^2} = -\frac{1}{z^2} \frac{1}{1-\frac{2}{z}} \underset{\left|\frac{2}{z}\right|<1}{=} \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{2^n}{z^n} =$$

$$= \sum_{n=2}^{\infty} \frac{2^{n-2}}{z^n} = \sum_{n=-\infty}^{-2} \frac{z^n}{2^{n+2}} \quad \text{valid in } D$$

$$f'(z) = g(z) = \sum_{n=-\infty}^{-2} \frac{n z^{n-1}}{2^{n+2}} = \sum_{n=-\infty}^{-3} \frac{(n+1)}{2^{n+3}} z^n$$

(2) Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$.
Calculate

$$\int_\gamma f(z)\, dz$$

when

a) $f(z) = z^3 e^{-1/z^4}$

b) $f(z) = (z^2 + z)\sin\left(\frac{1}{z}\right)$

Solution: a) $f(z) = z^3 e^{-1/z^4}$ is analytic in $\mathbb{C} \setminus \{0\}$. Let's expand in a Laurent series around 0. $e^w = \sum \frac{1}{n!} w^n$ so

$$f(z) = z^3 e^{-1/z^4} = z^3 \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n z^{-4n} =$$

$$= z^3 \left(1 - \frac{1}{z^4} + \frac{1}{2z^8} - \frac{1}{3!z^{12}} + \dots\right) =$$

$$= z^3 - \frac{1}{z} + \frac{1}{2z^5} - \dots$$

We see that

$\mathrm{Res}(0, f) = -1$

$$\Longrightarrow \int_\gamma z^3 e^{-1/z^4}\, dz = -2\pi i$$

b) We again need to identify singularities "inside" γ. $f(z) = (z^2+z)\sin\left(\frac{1}{z}\right)$ has a singularity at $z=0$ since $\sin\left(\frac{1}{z}\right)$ is undefined there. Now $\sin w = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n+1}}{(2n+1)!}$ so

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n-1} = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \cdots$$

and $$f(z) = (z^2+z)\left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \cdots\right)$$

$$= z + 1 - \frac{1}{6}\frac{1}{z} - \frac{1}{6}\frac{1}{z^2} + \cdots$$

That is $\mathrm{Res}(0,f) = -\frac{1}{6}$ and

$$\int_\gamma (z^2+z)\sin\left(\frac{1}{z}\right)dz = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}$$

3 Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+\pi^2)^2}\,dx$

(Hint: Evaluate $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{(z^2+\pi^2)^2}\,dz$ where $\gamma_R = [-R,R] + \tilde{\gamma}_R$ and $\tilde{\gamma}_R(t) = Re^{it}$, $0 \le t \le \pi$)

<u>Solution:</u>

$$e^{ix} = \cos x + i \sin x \quad \text{on } [-R, R]$$

and $\displaystyle\int_{-R}^{R} \frac{\sin x}{(x^2+\pi^2)^2} dx = 0$ since $\sin x$ is odd.

So $$\int_{-R}^{R} \frac{e^{ix}}{(x^2+\pi^2)^2} dx = \int_{-R}^{R} \frac{\cos x}{(x^2+\pi^2)^2} dx$$

We have $$\left| \int_{\tilde{\gamma}_R} \frac{e^{iz}}{(z^2+\pi^2)^2} dz \right| \le \int_{\tilde{\gamma}_R} \frac{1}{(R^2-\pi^2)^2} |dz| \le \frac{\pi R}{R^4}$$

(Real part of iz) ≤ 0 on $\tilde{\gamma}_R$
therefore $|e^{iz}| \le e^0 = 1$

$\dfrac{e^{iz}}{(z^2+\pi^2)^2}$ has singularities at $z_0 = \pm i\pi$ of order 2.

Let's calculate $\operatorname{Res}\left(i\pi, \dfrac{e^{iz}}{(z^2+\pi^2)}\right) =$

$$= \lim_{z \to i\pi} \frac{d}{dz} \left[(z-i\pi)^2 \frac{e^{iz}}{(z^2+\pi^2)^2} \right] =$$

$$= \lim_{z \to i\pi} \frac{d}{dz} \left(\frac{e^{iz}}{(z+i\pi)^2} \right) = \lim_{z \to i\pi} i \frac{e^{iz}}{(z+i\pi)^2} - 2 \frac{e^{iz}}{(z+i\pi)^3}$$

$$= i \frac{e^{-\pi}}{(2i\pi)^2} - \frac{2e^{-\pi}}{(2i\pi)^3} = e^{-\pi} \left(\frac{i}{-4\pi^2} - \frac{1}{4\pi^3(-i)} \right)$$

$$= i e^{-\pi} \left(-\frac{1}{4\pi^2} - \frac{1}{4\pi^3} \right) = -i \frac{e^{-\pi}}{4\pi^2} \left(1 + \frac{1}{\pi} \right)$$

So, since $\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{(z^2+\pi^2)^2} dz = 0$,

we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+\pi^2)^2} dx = 2\pi i \operatorname{Res}\left(i\pi, \frac{e^{iz}}{(z^2+\pi^2)^2}\right) =$$

$$= 2\pi i \, (-i) \frac{e^{-\pi}}{4\pi^2} \left(1+\frac{1}{\pi}\right) =$$

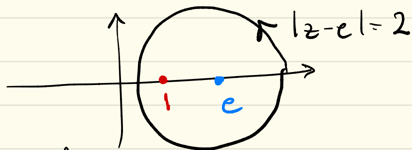
$$= \frac{e^{-\pi}}{2\pi^2}(1+\pi)$$

(if I didn't make calculation errors ☺)

(4) Calculate $\int_{|z-e|=2} \frac{1}{(z-1)\operatorname{Log}(z)} dz$

where $|z-e|=2$ is positively oriented.

Solution



The integrand has a singularity at $z_0=1$ (since both $(z-1)$ and $\operatorname{Log} z$ are zero there). Let check if the singularity is a pole. We check the order of the zero for $g(z) = (z-1)\,\mathrm{Log}(z)$.

$$g'(z) = \mathrm{Log}(z) + \frac{z-1}{z} \qquad g'(1) = \mathrm{Log}(1) + \frac{1-1}{1} = 0$$

$$g''(z) = \frac{1}{z} + \frac{1}{z^2} \qquad g''(1) = 1+1 = 2 \neq 0$$

$\Longrightarrow$ g has a zero of order 2 at $z_0 = 1$ and therefore $\frac{1}{(z-1)\mathrm{Log}(z)}$ has a pole of order 2 at $z_0 = 1$.

We calculate $\mathrm{Res}\left(1, \frac{1}{(z-1)\mathrm{Log}(z)}\right) =$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{(z-1)^2}{(z-1)\mathrm{Log}(z)}\right)$$

$$\frac{d}{dz}\frac{(z-1)}{\mathrm{Log}(z)} = \frac{\mathrm{Log}(z) - \frac{z-1}{z}}{[\mathrm{Log}(z)]^2} = \frac{z\,\mathrm{Log}(z) - z + 1}{z(\mathrm{Log}(z))^2}$$

l'Hospital rule

$$\lim_{z\to 1} \frac{z\,\mathrm{Log}(z) - z + 1}{z(\mathrm{Log}(z))^2} = \lim_{z\to 1} \frac{\mathrm{Log}(z) + \frac{z}{z} - 1}{\mathrm{Log}(z)^2 + \frac{2z}{z}\mathrm{Log}(z)} =$$

$$= \lim_{z\to 1} \frac{1}{\mathrm{Log}(z) + 2} = \frac{1}{2}$$

$$\int_{|z-e|=2} \frac{1}{(z-1)\,\mathrm{Log}(z)}\,dz = 2\pi i\,\mathrm{Res}\left(1, \frac{1}{(z-1)\,\mathrm{Log}(z)}\right) =$$

$$= 2\pi i \cdot \frac{1}{2} = \pi i$$