ELEC-E8116 Model-based control systems
/exercises 11 Solutions

Problem 1. Consider the general system representation

$$\dot{x} = Ax + Bu + Nw$$
$$z = Mx + Du$$
$$y = Cx + w$$

where it is assumed that

$$D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

Show that this assumption can be relaxed by taking

$$\tilde{u} = \left(D^T D\right)^{1/2} u + \left(D^T D\right)^{-1/2} D^T Mx$$

and

$$z = \tilde{M}x + \tilde{D}\tilde{u}, \quad \tilde{M} = \left(I - D\left(D^T D\right)^{-1} D^T\right)M, \quad \tilde{D} = D\left(D^T D\right)^{-1/2}$$

Solution. First we show that the value of z remains the same:

$$\tilde{M}x + \tilde{D}\tilde{u} = \left[I - D\left(D^T D\right)^{-1} D^T\right]Mx + D\left(D^T D\right)^{-1/2} \cdot \left\{\left(D^T D\right)^{1/2} u + \left(D^T D\right)^{-1/2} D^T Mx\right\}$$

$$= Mx - D\left(D^T D\right)^{-1} D^T Mx + D\left(D^T D\right)^{-1/2}\left(D^T D\right)^{1/2} u + D\underbrace{\left(D^T D\right)^{-1/2}\left(D^T D\right)^{-1/2}}_{\left(D^T D\right)^{-1}} D^T Mx$$

$$= Mx - D\left(D^T D\right)^{-1} D^T Mx + D\underbrace{\left(D^T D\right)^{-1/2}\left(D^T D\right)^{1/2}}_{I} u + D\left(D^T D\right)^{-1} D^T Mx$$

$$= Mx - \cancel{D\left(D^T D\right)^{-1} D^T Mx} + Du + \cancel{D\left(D^T D\right)^{-1} D^T Mx}$$

$$= Mx + Du = z$$

Ok.

Then it remains to show that $\tilde{D}^T \begin{bmatrix} \tilde{M} & \tilde{D} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

$\tilde{D}^T \begin{bmatrix} \tilde{M} & \tilde{D} \end{bmatrix} = \begin{bmatrix} \tilde{D}^T \tilde{M} & \tilde{D}^T \tilde{D} \end{bmatrix}$. Let us consider both submatrices separately

Fist submatrix

$$\tilde{D}^T \tilde{M} = \left(D\left(D^T D\right)^{-1/2} \right)^T \left(I - D\left(D^T D\right)^{-1} D^T \right) M$$

$$= \left(D^T D\right)^{-T/2} D^T \left(I - D\left(D^T D\right)^{-1} D^T \right) M$$

$$= \left(D^T D\right)^{-1/2} D^T \left(I - D\left(D^T D\right)^{-1} D^T \right) M$$

$$= \left(D^T D\right)^{-1/2} D^T M - \left(D^T D\right)^{-1/2} \underbrace{D^T D\left(D^T D\right)^{-1}}_{I} D^T M$$

$$= \left(D^T D\right)^{-1/2} D^T M - \left(D^T D\right)^{-1/2} D^T M = 0$$

Second submatrix

$$\tilde{D}^T \tilde{D} = \left(D^T D\right)^{-T/2} D^T D \left(D^T D\right)^{-1/2}$$

$$= \left(D^T D\right)^{-1/2} D^T D \left(D^T D\right)^{-1/2}$$

$$= \left(D^T D\right)^{-1/2} \left(D^T D\right)^{1/2} = I$$

Ok.

Note that in above $\left(D^T D\right)^{-T/2} = \left(\left(D^T D\right)^{-1/2} \right)^T = \left(\left(D^T D\right)^T \right)^{-1/2} = \left(D^T D\right)^{-1/2}$. Remember the rules of matrix calculus.

Problem 2. The *Frobenius norm* of a matrix A is defined as

$$\|A\|_F = \sqrt{\sum_{i,j} \left|a_{ij}\right|^2} = \sqrt{\text{tr } (A^* A)}$$

Show that $\|A\|_F = \sqrt{\sum_i \sigma_i^2(A)}$, where $\sigma_i(A)$ are the singular values of A.

Solution. We do the singular value decomposition of matrix A

$$A = U\Sigma V^* \Rightarrow A^* A = V\Sigma^T \overbrace{U^* U}^{I} \Sigma V^* = V\Sigma^T \Sigma V^*$$

But generally (see Exercise 2) it holds tr(CD) = tr (DC), provided that the dimensions agree of course. Then

$$\text{tr } (A^* A) = \text{tr } (V\Sigma^T \Sigma V^*) = \text{tr } (\Sigma V^* V \Sigma^T)$$
$$= \text{tr } (\Sigma\Sigma^T) = \sum_i \sigma_i^2(A)$$