ELEC-E8116 Model-based control systems
/exercises 12 solutions

Problem 1. Consider a SISO system in a two-degrees-of-freedom control configuration. Let the loop transfer function be $L(j\omega)=G(j\omega)F_y(j\omega)$, where the symbols are standard used in the course.

a. Define the *sensitivity* and *complementary sensitivity functions* and determine where in the complex plane it holds

$$|S(j\omega)|<1, \quad |S(j\omega|=1, \quad |T(j\omega)|<1 \text{ and } |T(j\omega)|=1$$

b. Let the Nyquist diagram of the loop transfer function approach from below the point where $|S(j\omega_n|=1$ and assume that it also holds then $|T(j\omega_n|=1$. Assuming that there are no right half poles of the open loop transfer function, what is the phase margin of the closed-loop system? Hint. In the complex plane (xy) let $L(j\omega)=x(\omega)+jy(\omega)$.

Solution.

a. Standard definitions, see lecture slides, Chapter 3. In the SISO case

$$L(j\omega)=G(j\omega)F_y(j\omega)$$
$$S(j\omega)=\frac{1}{1+L(j\omega)}$$
$$T(j\omega)=\frac{L(j\omega)}{1+L(j\omega)}$$

Denote $L(j\omega)=x(\omega)+jy(\omega)$ and calculate

$$S=\frac{1}{1+x+jy}\Rightarrow|S|=\frac{1}{\sqrt{(1+x)^2+y^2}}\Rightarrow(1+x)^2+y^2=\frac{1}{|S|^2}$$

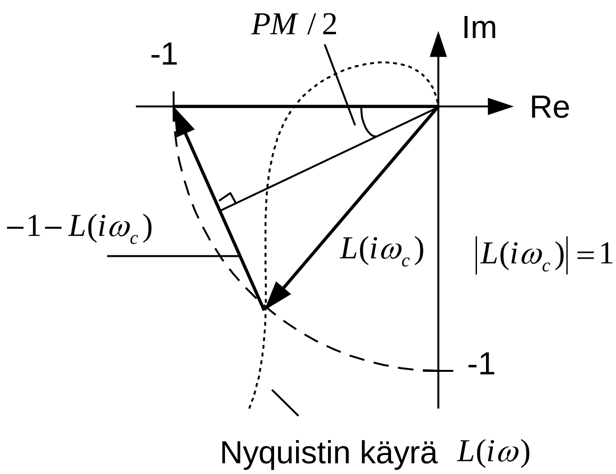
In the complex (*x*-*y*) plane this is a circle with the center point (-1,0) and radius $1/|S|$. Consider the circle with radius 1. On the circle $|S|=1$, outside the circle $|S|<1$, inside the circle $|S|>1$. So when the Nyquist diagram of $L(j\omega)$ enters the circle from outside to inside the absolute value of *S* obtains the above values accordingly.

Now $T=\dfrac{L}{1+L}=\dfrac{x+iy}{1+x+iy}\Rightarrow |T|=\dfrac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}=\sqrt{\dfrac{x^2+y^2}{(1+x)^2+y^2}}=\sqrt{\dfrac{x^2+y^2}{x^2+y^2+2x+1}}$

Clearly
$$|T|=1\Rightarrow 2x+1=0,\Rightarrow x=-1/2$$
$$|T|<1\Rightarrow 2x+1>0,\Rightarrow x>-1/2$$

The absolute value of T is 1 on the line x=-1/2 on the complex plane. $|T|<1$ holds for all points to the right of this line.

b. We look at the figure



Consider the dashed circle. The Nyquist curve of L crosses this circle at $|L(j\omega_c|=1$. But we know that $|S(j\omega_n|=|T(j\omega_n|=1$, so $\omega_n=\omega_c$ (the gain crossover frequency). Based on part a. we know that on the line Re(-1/2) the value of $|T|$ is 1. Therefore the circle $|S|=1$ (see part a.) intersects the dashed circle $|L(j\omega|=1$ exactly at the point given by the vector $|L(j\omega_c|=1$. We have an equilateral triangle (see figure), where the angles are 60 degrees.

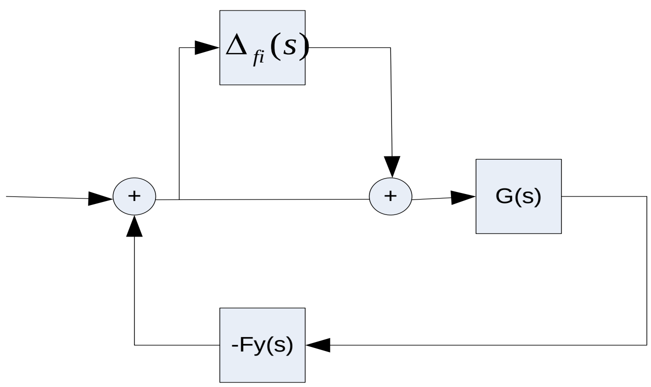
The same result could have been obtained by considering the right triangle with one cathetus ½, hypotenuse 1 and the angle PM between them.

The assumption of no RHP poles in the L function was needed to guarantee stability (and hence positive phase margin) when the Nyquist curve does not enclose the critical point (-1,0).

Problem 2. You are given the nominal plant

$$G(s) = \frac{10}{s^2 + 4}$$

with an input feedback uncertainty $\|\Delta_{fi}(s)\|_\infty \le 0.5$, and the controller $F_y(s) = \frac{4(s+2)}{s+8}$ (see Fig.) What can be said about robust stability of the closed-loop system?



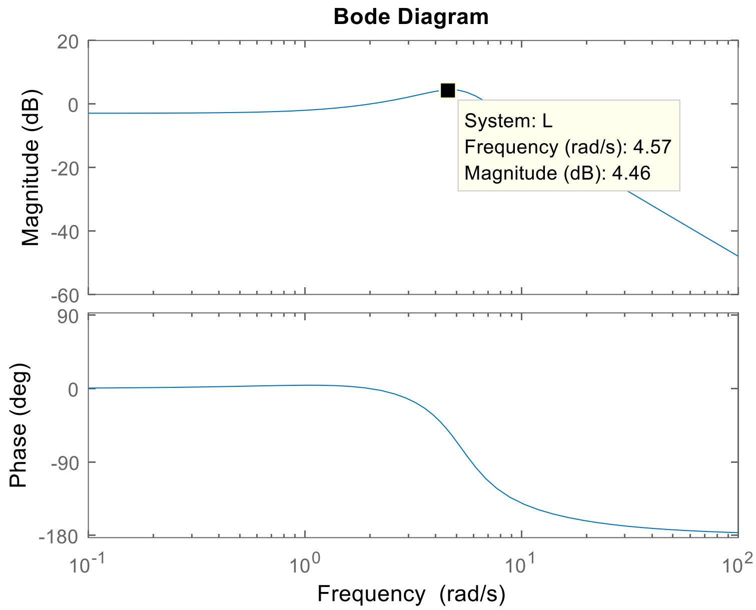
Solution. We have the case with multiplicative uncertainty discussed in Lectures, Chapter 3 ("Robustness"). (See however a note in the end of the solution.) As for the Small Gain Theorem see Chapter 1.

The condition for robust stability is

$|T| < \frac{1}{|\Delta_{fi}|}$. We know that $|\Delta_{fi}(j\omega)| \le 0.5$ for all frequencies. Therefore the condition for robust stability in this case becomes

$$|T| < 2 \text{ or } 20\lg(2)\text{dB} \approx 6 \text{ dB}$$

Calculate $T = \frac{L}{1+L} = \frac{GF_y}{1+GF_y} = \cdots = \frac{40(s+2)}{(s+4)(s^2+4s+28)}$. The Bode diagram is shown in the figure. The maximum peak is about 4,5 dB, so the system is robustly stable. By Matlab: *hinfnorm*(*T*) = 1.6797 or 4.5046 dB.



Note: The solution is correct, because the system is a SISO case. But in the lectures the multiplicative uncertainly was defined as $G_0 = (I + \Delta_G)G$, which is not exactly as in the figure of the problem (nominal plant G should be in front of the uncertainty branch). So actually the result $|T| < \dfrac{1}{|\Delta_{fi}|}$ would in the MIMO case not hold (what would the condition for robust stability be in this case?).

Problem 3. Consider a SISO system and a state feedback control

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$u(t) = -Lx(t)$$

where L is chosen as a solution to the infinite time optimal (LQ) horizon problem.

a. Prove that the loop gain is $H(s) = L(sI - A)^{-1}B$

b. Prove that $|1 + H(i\omega)| \geq 1$

c. Show that for the LQ controller

- phase margin is at least 60 degrees
- gain margin is infinite
- the magnitude of the sensitivity function is less than 1
- the magnitude of the complementary sensitivity function is less than 2.

Solution:

a. First solve for x: $px = Ax + Bu \Rightarrow x = [pI - A]^{-1} Bu$

Starting from the output of the controller u go around the loop and meet the signal u again. We get

$$u = -Lx = -L[pI - A]^{-1} Bu$$

The open loop transfer function is the forward loop transfer function multiplied by the feedback loop transfer function. The open loop is then

$$H(s) = L[sI - A]^{-1} B$$

as given in the problem. Note: no minus sign, because it is the feedback sign.

b. In the LQ problem

$H(s) = L[sI - A]^{-1} B$ Note that L is now the state feedback gain, H is the open loop transfer function.

The (stationary) Riccati equation: $A^T S + SA + Q - SBR^{-1}B^T S = 0$.
State feedback gain: $L = R^{-1}B^T S$.

In the exercise session the problem was solved in the simple case of assuming one-dimensional state variable x. Then all the matrices are scalars:

$$\left|1 + H(j\omega)\right|^2 = (1 + H(j\omega))^*(1 + H(j\omega)) = (1 + H(-j\omega))(1 + H(j\omega))$$

$$= \left(1 + \frac{lb}{-j\omega - a}\right)\left(1 + \frac{lb}{j\omega - a}\right) = \frac{-a + lb - j\omega}{-a - j\omega} \cdot \frac{-a + lb + j\omega}{-a + j\omega}$$

$$= \frac{(-a + lb)^2 + \omega^2}{a^2 + \omega^2} = \frac{a^2 - 2abl + b^2 l^2 + \omega^2}{a^2 + \omega^2}$$

$$= \frac{a^2 - 2a\frac{b^2}{r}s + b^2\frac{b^2 s^2}{r^2} + \omega^2}{a^2 + \omega^2} = \frac{a^2 + \frac{b^2}{r}(\frac{b^2 s^2}{r} - 2as) + \omega^2}{a^2 + \omega^2}$$

$$= \frac{a^2 + \frac{b^2}{r}q + \omega^2}{a^2 + \omega^2} \geq 1$$

because $\frac{b^2}{r}q \geq 0$. Note how the Riccati equation was used in the last part of the derivation.

But the general inequality is

$$[I + H(-j\omega)]^T R[I + H(j\omega)] \geq R$$

which applies also to multivariable cases. In the case of single transfer functions the above trivially simplifies to

$$|1 + H(i\omega)| \geq 1$$

The general proof (MIMO case) is however a bit more complicated.

$$\begin{aligned}
&[I + H(-j\omega)]^T R[I + H(j\omega)] = [I + H(-j\omega)]^T [R + RH(j\omega)] \\
&= R + RH(j\omega) + H(-j\omega)^T R + H(-j\omega)^T RH(j\omega) \\
&= R + RL[j\omega I - A]^{-1} B + B^T [-j\omega I - A]^{-T} L^T R + B^T [-j\omega I - A]^{-T} L^T RL[j\omega I - A]^{-1} B \\
&= R + B^T S[j\omega I - A]^{-1} B + B^T \left[-j\omega I - A^T\right]^{-1} SB + B^T \left[-j\omega I - A^T\right]^{-1} SBR^{-1}B^T S[j\omega I - A]^{-1} B \\
&= R + B^T \left[-j\omega I - A^T\right]^{-1} \left\{\left[-j\omega I - A^T\right]S + S[j\omega I - A] + SBR^{-1}B^T S\right\}[j\omega I - A]^{-1} B \\
&= R + B^T \left[-j\omega I - A^T\right]^{-1} \left\{-A^T S - SA + A^T S + SA + Q\right\}[j\omega I - A]^{-1} B \\
&= R + B^T \left[-j\omega I - A^T\right]^{-1} Q[j\omega I - A]^{-1} B \geq R
\end{aligned}$$

To see the last inequality note that *R* is positive definite. The matrix

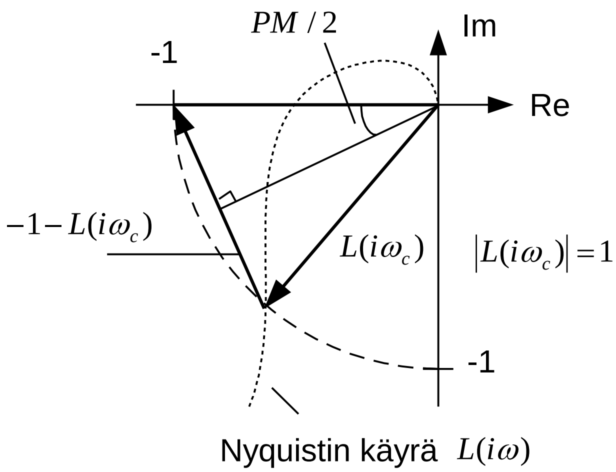
$$Z = B^T \left[-j\omega I - A^T\right]^{-1} Q[j\omega I - A]^{-1} B$$

is clearly real, because $Z^* = Z$ (the matrix is in fact Hermitian). But for any non-zero vector *x* with appropriate dimension

$$\begin{aligned}
&x^* Z x = x^* B^T \left[-j\omega I - A^T\right]^{-1} Q[j\omega I - A]^{-1} Bx \\
&= \left[(j\omega I - A)^{-1} Bx\right]^* Q\left[(j\omega I - A)^{-1} Bx\right] = y^* Q y \geq 0
\end{aligned}$$

Hence *Z* is positive semidefinite. Note that *Q* and *R* are positive definite by definition.

c. Consider the following figure, where $L = H$ now is the loop transfer function.



Because $|1+H(i\omega)| \geq 1$ the Nyquist curve will never enter inside the circle centered at (-1,0) and with the radius 1. Therefore the gain margin is infinite and the sensitivity function is never larger than 1 in magnitude. The complementary sensitivity function cannot be larger than 2, because the two sentitivity functions can differ at most by 1 in magnitude. Now the Nyquist curve touches the dashed line at the gain crossover frequency ω_c and if $|1+L(i\omega)| = 1$ (minimum) we have an equilateral triangle (see figure) so that each angle is 60 degrees. But generally $|1+L(i\omega)| \geq 1$ so that the phase margin is at least 60 degrees.