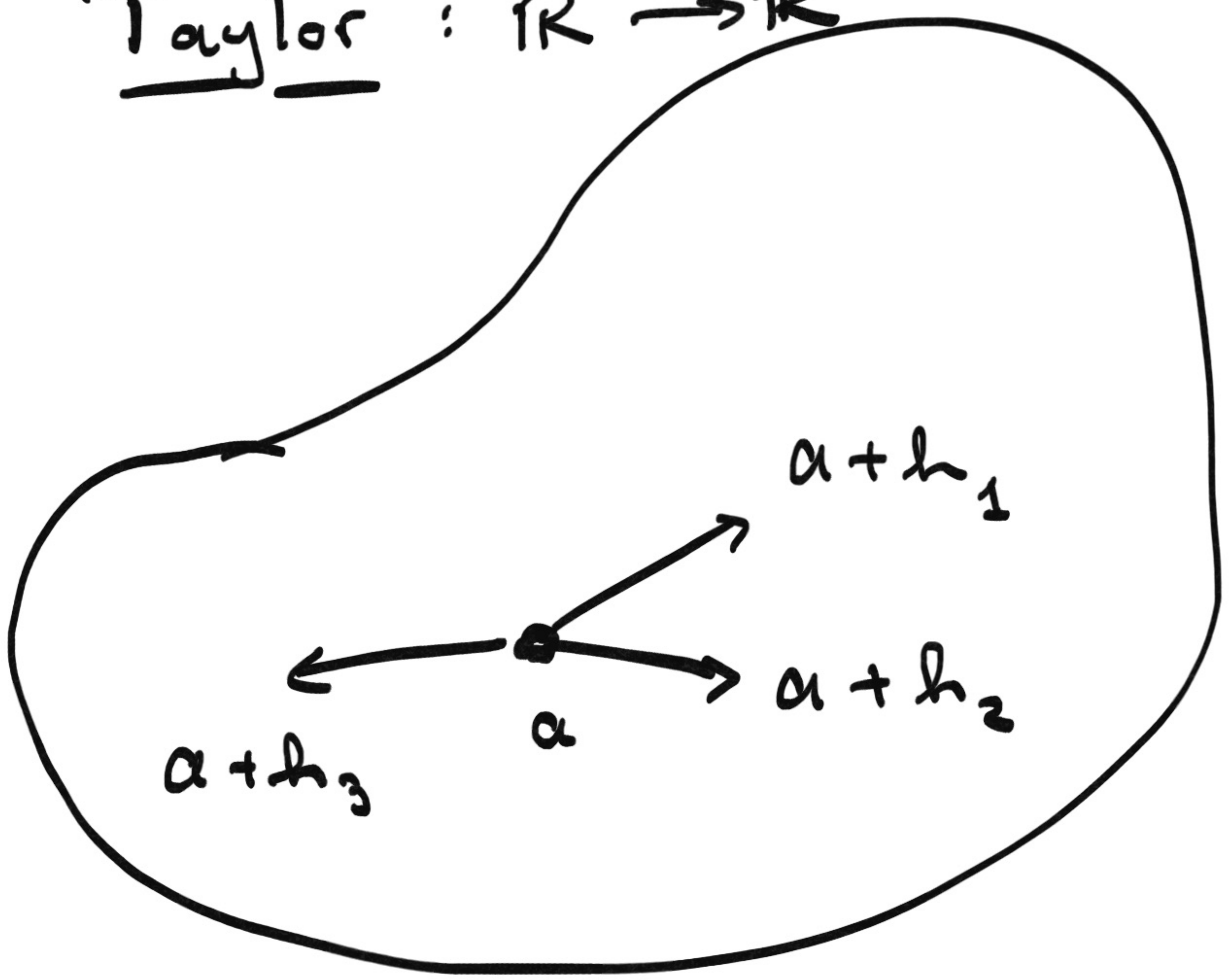
# ÄÄRIARVOJEN LUOKITTELU

<u>Taylor</u> : $\mathbb{R}^n \to \mathbb{R}$




Jatkuvat osittaisderivaatat

janoille $a \to a + h_i$

$$f(a+h) \simeq \sum_{j=0}^{m} \frac{(h^T \nabla)^j f(a)}{j!}$$

Kriittisessä pisteessä $x$ : $\nabla f(x) = 0$

Toisen asteen Taylor:

$$f(x+h) \simeq f(x) + \cancel{h^T \nabla f(x)} + \frac{1}{2} (h^T \nabla)^2 f(x)$$

$$\Rightarrow f(x+h) - f(x) \simeq \frac{1}{2} (h^T \nabla)^2 f(x)$$

Edellisen luennon esimerkki:

Korkeammat derivaatat:

$$\frac{1}{2} \Big( h^2 f_{11}(a,b) + hk f_{12}(a,b) + kh f_{21}(a,b) + k^2 f_{22}(a,b) \Big)$$

$$= \frac{1}{2} \underbrace{\begin{pmatrix} h \\ k \end{pmatrix}^T}_{1 \times 2} \underbrace{\underbrace{\begin{pmatrix} f_{11}(a,b) & f_{21}(a,b) \\ f_{12}(a,b) & f_{22}(a,b) \end{pmatrix}}_{2 \times 2}}_{\text{Hessen matriisi } H_f(\underline{x})} \underbrace{\begin{pmatrix} h \\ k \end{pmatrix}}_{2 \times 1}$$

$$H_f(\underline{x}) = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} f(\underline{x}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f(\underline{x}) \\ \frac{\partial^2}{\partial x_1 \partial x_2} f(\underline{x}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} f(\underline{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_n} f(\underline{x}) & \cdots & \frac{\partial^2}{\partial x_n^2} f(\underline{x}) \end{pmatrix}$$

Hessen matriisi on symmetrinen!

<u>Neliömuoto</u> $x^T A x$

$\underset{n \times n}{A}$ on symmetrinen : $A = A^T$

Reaalisen symmetrisen matriisin ominaisarvot ovat reaalisia.

$\lambda_i > 0$ : positiivisesti definiitti

$\lambda_i < 0$ : negatiivisesti definiitti

$\lambda_i > 0$ ja $\lambda_j < 0$ : indefiniitti
$i \neq j$

Jos $A$ on positiivisesti definiitti, niin $x^T A x > 0$, kaikilla $x \in \mathbb{R}^n$.

Merkitään : pos. def.

$A$ on diagonalisoituva :

$A = Q \Lambda Q^T$, $Q$ ortogonaalinen

$Q = (v_1 \; v_2 \; \dots \; v_n)$

Mielivaltainen $y = \sum_{i=1}^{n} \alpha_i v_i$

$\Rightarrow y^T A y =$
$\alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \dots + \alpha_n^2 \lambda_n$

$\Rightarrow y^T A y > 0$ vain jos $\lambda_i > 0$

## Sylvesterin kriteeri

$$A = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & & \vdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & & \\ \vdots & & & \ddots & \\ \alpha_{n1} & & \cdots & & \alpha_{nn} \end{vmatrix}$$

$$\Delta_1 = |\alpha_{11}|$$

$$\Delta_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}$$

$\Delta_3 =$ jne.

A on pos. def. jos kaikille $k = 1, 2, \ldots, n$ : $\Delta_k > 0$

A on neg. def. jos merkit alternoivat : $\det(-A) = (-1)^n |A|$

## Esimerkki

$$A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 4 \end{pmatrix}$$

$$\Delta_1 = 4$$

$$\Delta_2 = 4 \cdot 2 - (-1 \cdot -1) = 7$$

$$\Delta_3 = 32 - 2 - 2 - 2 - 16 - 4 = 6$$

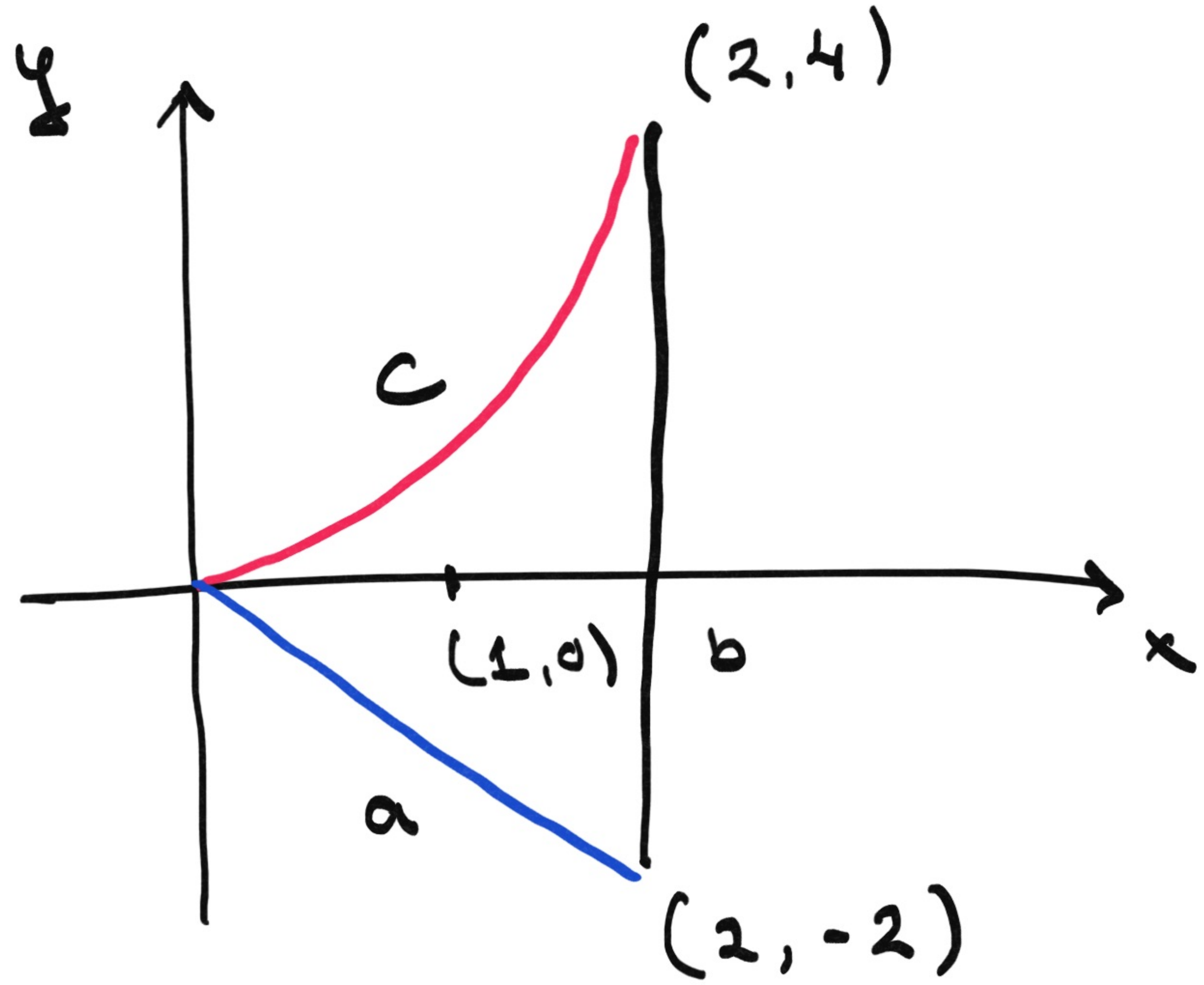
A on pos. def.

## Ääriarvoesimerkki

$$f(x,y) = xy - y$$

$$A = \{(x,y) \mid x \in [0,2], -x \leq y \leq x^2\}$$

Etsitään maksimi ja minimi.




(i) kriittiset pisteet:

$$\begin{cases} f_x = y = 0 \\ f_y = x - 1 = 0 \end{cases}$$

kp $(1,0) \in A$

$f(1,0) = 0$

(ii) Reunakomponentit:

$a = \{(x,y) \mid x \in [0,2],\ y = -x\}$

$g^{(a)}(x) = -x^2 + x$

$Dg^{(a)}(x) = -2x + 1 = 0$

$\Rightarrow x = +\frac{1}{2},\ y = -\frac{1}{2}$

Päätepisteet: $g^{(a)}(0) = 0$

$g^{(a)}(2) = -2$

$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

b: $g^{(b)}(y) = 2y - y = y$

$g^{(b)}(-2) = -2$

$g^{(b)}(4) = 4$

c: $g^{(c)}(x) = x^3 - x^2$

$Dg^{(c)}(x) = 3x^2 - 2x = 0$

$\Leftrightarrow x = 0$ tai $x = \frac{2}{3}$

$f\left(\frac{2}{3}, \frac{4}{9}\right) = -\frac{4}{27}$

Mahdollisten arvojen joukko:

$\left\{ 0, \frac{1}{4}, 0, -2, 4, -\frac{4}{27} \right\}$

$\max_A f = 4$

$\min_A f = -2$

$f(2,4) = 2 \cdot 4 - 4 = 4$