Lecture V - NP Problems and Graphs

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Previously on..

Matching Problems:

- Weighted Matching;
- Maximum Matching.

PREVIOUSLY ON...

So far, all problems either have **algorithm** or **a mixed integer** formulation.

But what happens when **an algorithm** is not achievable nor efficient? A mixed integer formulation **might still** be possible, but would it be enough?

Complexity class P

Decision Problems

Decision problem is a yes-or-no problem.

They differ from **optimization problem**, because the former requires an answer that have an **optimal configuration**.

For instance: "*What is the shortest path between two nodes?*" **vs** "*Is a particular path P the shortest path between these two nodes?*".

Remark: An optimization problem has a **corresponding** decision problem.

Class P

- P is the **class** of all decision problems (X, Y) for which there is a **polynomial** time algorithm.
- Given $x \in A^*$: compute $f(x) \in \{0, 1\}$ with time$(x) \leq p($size$(x))$.

Examples

- linear inequalities;
- shortest path;
- maximum matching;
- minimum cost flow.
- ...

Complexity class NP

Class NP

- decision problem (X, Y) belongs to class NP if for each $y \in Y$ a certificate c can be verified in **polynomial time**
- usually c is a **feasible solution** to the problem
- name $NP =$ **nondeterministic** polynomial: "guessing" certificates long enough would work
- (X, Y) can be solved by **nondeterministic in polynomial time**

Examples

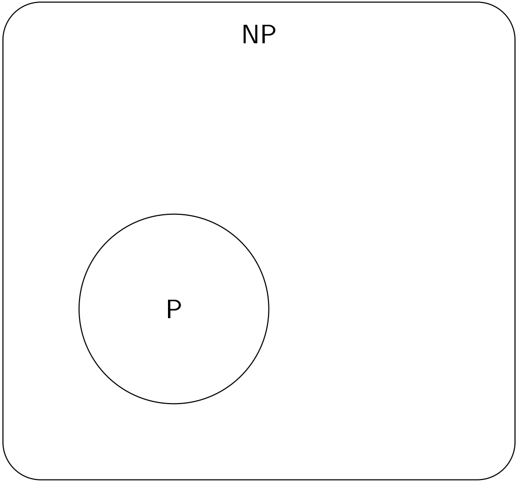
- integer linear inequalities $\rightarrow c$ is feasible vector x
- knapsack $\rightarrow c$ is a feasible set of items to take

P vs. NP

Theorem
$P \subseteq NP$.

Proof.
$(X, Y) \in P$ can be decided in polynomial time. $\Rightarrow x$ can be used as **certificate**. $\square$

P vs. NP



TSP - Travelling Salesman Problem

Definition

TSP: Travelling Salesman (or Salesperson) Problem

Imagine a scenario, where a set of cities are expecting a visit from a **travelling** merchant.

As part of their visit, this **salesman** has to start and finish their travel in the same city, cannot visit the same city more than once and every city has to be visited in a single trip.

Example

Figure: A lone traveller about to make important decisions

→ What is the shortest possible route?

History

Such problem is called Travelling Salesman Problem, very important to the fields of theoretical computer science and operations research.

It was first described by Irish mathematician **W.R. Hamilton** and British mathematician **Thomas Kirkman** in the 1800s through the description of a game where the solution involved a cycle without overlapping nodes.

Initial Approach

At first glance, the first solution is to try all possibilities and choose the best solution.

$\longrightarrow$ enumeration process

Initial challenge: for an instance with n cities, there are 2^n possible combinations.

$\longrightarrow$ impractical.

Is there any **better alternative**?

Modelling - Graph

First, let us assume that the set of cities can be modelled as graph $G = (V, E, f)$, where:

- V is the set of individual cities;
- E represent the paths between a pair of cities and;
- f_{ij} is the cost to travel from city i to city j, for all $(i, j) \in E$.

Modelling - Variable and Objective

The choice to travel from city i to city j using a path (edge in our modelling) connecting them is modelled by our decision variable.

$$x_{ij} = \left\{ \begin{array}{ll} 1, & \text{if path goes from city } i \text{ to city } j \\ 0, & \text{otherwise} \end{array} \right\}$$

Our objective function can be also derived from the problem description:

$$\min \sum_{i}^{n} \sum_{j, j \neq i}^{n} x_{ij} f_{ij}$$

where n is the number of cities ($|V| = n$).

Modelling - Constraints

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Two constraints can also be derived directly from description:

Singular incoming degree:

$$\sum_{i=1, i \neq j}^{n} x_{ij} = 1 \quad \forall j \in \{1, \cdots, n\}$$

Singular outgoing degree:

$$\sum_{j=1, j \neq i}^{n} x_{ij} = 1 \quad \forall i \in \{1, \cdots, n\}$$

Those constraints are characterize **paths**.

Modelling - Split version

These constraints imposed that every city is visited only once.

However, they do not guarantee that there is a single trip will connecting all cities.

For instance:

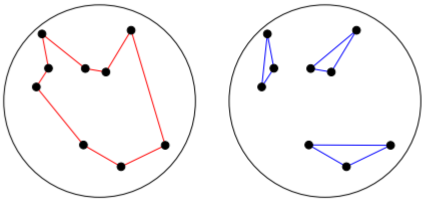

Figure: Two solutions that do not violate the previous constraints, but only one has a single trip.

Modelling Decisions - MTZ vs DFJ

There are two main strategies to prevent separate tour (subtour) from our potential solution: Miller-Tucker-Zemlin and Dantzig-Fulkerson-Johnson.

Both impose the presence of a single tour using **linear constraints**.

Miller-Tucker-Zemlin

Requires an additional variable to track which city has been visited starting from initial city $i = 1$.

$\longrightarrow$ By setting $u_j > u_i$, it determines the **order** of visiting each city (city j will be visited after city i).

Miller-Tucker-Zemlin

This leads to following requirement:

$$u_j \geq u_i + 1 \text{ if } x_{ij} = 1$$

which can encapsulated as the following constraints:

$$\begin{aligned} & u_i - u_j + 1 \leq (n-1)(1 - x_{ij}) && \text{for } 2 \leq i \neq j \leq n \\ & 2 \leq u_i \leq n && \text{for } 2 \leq i \leq n \end{aligned}$$

Miller-Tucker-Zemlin

Minimize $\sum_{i}^{n} \sum_{j, j \neq i}^{n} x_{ij} f_{ij}$

Subject to:

$$\sum_{i=1, i \neq j}^{n} x_{ij} = 1 \qquad \forall j \in \{1, \cdots, n\}$$

$$\sum_{j=1, j \neq i}^{n} x_{ij} = 1 \qquad \forall i \in \{1, \cdots, n\}$$

$$u_i - u_j + 1 \leq (n-1)(1 - x_{ij}) \qquad 2 \leq i \neq j \leq n$$

$$2 \leq u_i \leq n \qquad 2 \leq i \leq n$$

$$x_{ij} \in \{0, 1\} \qquad i, j \in \{1, \cdots, n\}$$

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Dantzig-Fulkerson-Johnson

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Impose an extra requirement that eliminates all subset of nodes to create a subtour.

$$\sum_{i \in Q} \sum_{j \neq i, j \in Q} x_{ij} \leq |Q| - 1 \qquad\qquad \forall Q \subset \{1, \cdots, n\}, |Q| \geq 2$$

Dantzig-Fulkerson-Johnson

Minimize $\sum_{i}^{n} \sum_{j,j \neq i}^{n} x_{ij} f_{ij}$

Subject to:

$$\sum_{i=1,i\neq j}^{n} x_{ij} = 1 \qquad \forall j \in \{1, \cdots, n\}$$

$$\sum_{j=1,j\neq i}^{n} x_{ij} = 1 \qquad \forall i \in \{1, \cdots, n\}$$

$$\sum_{i \in Q} \sum_{j \neq i, j \in Q} x_{ij} \leq |Q| - 1 \qquad \forall Q \subset \{1, \cdots, n\}, |Q| \geq 2$$

$$x_{ij} \in \{0, 1\} \qquad i, j \in \{1, \cdots, n\}$$


thank you