Combinatorial Optimization

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NP Problems and Polynomial Transformation

§ Week VII §

Problem 1: NP Problems and Aliens

Highly intelligent aliens land on Earth and tell us the following two things and then leave immediately.

1. The 3-Coloring problem (which is NP-complete) is solvable in worst-case $O(n^9)$ time, where n denotes the number of vertices in the graph.

2. There is no algorithm for 3-Coloring that runs faster than $\Omega(n^7)$ time in the worst case.

Assuming these two facts, for each of the following assertions, indicate whether it can be inferred from the information the aliens have given us. (In all cases, time complexities are understood to be worst-case running time.) Provide a short justification in each case.

- All NP-complete problems are solvable in polynomial time.
- All problems in NP, even those that are not NP-complete, are solvable in polynomial time.
- All NP-hard problems are solvable in polynomial time.
- All NP-complete problems are solvable in $O(n^9)$ time.
- No NP-complete problem can be solved faster than $\Omega(n^7)$

Solution:

- All NP-complete problems are solvable in polynomial time: Yes. Every problem in NP is polynomially reducible to SAT, and SAT is reducible to every NP-hard problem. Therefore, a polynomial time solution to any NP-hard problem (such as 3Col) implies that every problem in NP can be solved in polynomial time. Since the set of NP-complete problems is a subset of NP, it follows that they are all solvable in polynomial time.
- All problems in NP, even those that are not NP-complete, are solvable in polynomial time: Yes, for the same reason given part (a).
- All NP-hard problems are solvable in polynomial time: This does not follow. Indeed, there may NP-hard problems that are much harder than problems in NP.
- All NP-complete problems are solvable in $O(n^9)$ time: This does not follow. Transformations may increase the size of the input of a problem (by a polynomial amount). Thus, if a reduction from some other NP-complete problem, call it X, to 3Col, expands an input of size n into an input of size n^2, then using this reduction to solve problem X solution could take as much time as $O((n^2)^) = O(n^{18})$ time. This is still polynomial, but not the same as $O(n^9)$.
- No NP-complete problem can be solved faster than $\Omega(n^7)$ time in the worst case: This does not follow. This no correlaction or guarantee for the performance of any solution involving NP-Hard problems.

Problem 2: Hamiltonian Path

Given an undirected graph $G = (V, E)$, a Hamiltonian path is a simple path (not a cycle) that visits every vertex in the graph. The Hamiltonian Path problem (HP) is the problem of determining whether a given graph has a Hamiltonian path.

1. Show that HP is in NP.
2. Professor Gwen Stacy observes that if a graph has a Hamiltonian Cycle, then it also has a Hamiltonian Path. He suggests the following trivial reduction in order to prove that HP is NP-hard. Given a graph G for the Hamiltonian Cycle problem, simply output a copy of this graph. Explain why Professor Stacy's reduction is incorrect.
3. Give a (correct) proof that HP is NP-hard. (Hint: The reduction is from the Hamiltonian Cycle problem or Travelling Salesman Person)

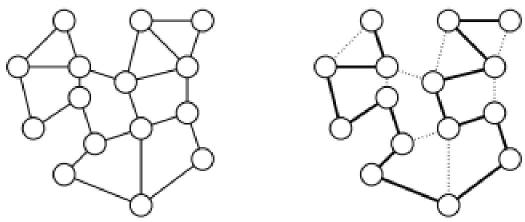

Figure 1: Hamiltonian path

Solution:

- 2 steps to proof any NP Problem: verify the feasibility of a solution using in polynomial time and find a "transformation" into another NP problem. First, an easy way to verify solution is to check easy node in the path and check whether the actual path can exist in the graph. Second, if you remove the weight cost condition from TSP, the resulting problem is essentially Hamiltonian Cycle.

  Alternatively:

  Hamiltonian Cycle is NP-Hard In order to prove the Hamiltonian Cycle is NP-Hard, we will have to reduce a known NP-Hard problem to this problem. We will reduce from the Hamiltonian Path problem to the Hamiltonian Cycle problem. Every instance of the Hamiltonian Path problem consisting of a graph G =(V, E) as the input can be converted to a Hamiltonian Cycle problem consisting of graph G' = (V', E'). We will construct the graph G' in the following way:

  - V': Add vertices V of the original graph G and vertex Vnew such that all the vertices connected to the graph are connected to this vertex. The number of vertices increases by 1, $V' = V + 1$.
  - E': Add edges E of the original graph G and add new edges between the newly added vertex and the original vertices of the graph. The number of edges increases by the number of vertices V that is, $E' = E + V$.
  - Let us assume that the graph G contains a Hamiltonian path covering the V vertices of the graph starting at a random vertex, say Vstart, and ending at Vend, now since we connected all the vertices to an arbitrary new vertex Vnew in G'. We extend the original Hamiltonian Path to a Hamiltonian Cycle using the edges Vend to Vnew and Vnew to Vstart, respectively. Graph G' now contains the closed cycle traversing all vertices once.

  - We assume that the graph G' has a Hamiltonian Cycle passing through all the vertices, including Vnew. We remove the edges corresponding to the vertex Vnew in the cycle to convert it to a Hamiltonian Path. The resultant path will cover the vertices V of the graph and will cover them exactly once.

  Thus, we can say that graph G' contains a Hamiltonian Cycle if graph G contains a Hamiltonian Path. Therefore, any instance of the Hamiltonian Cycle problem can be reduced to one of the Hamiltonian Path problems. Thus, the Hamiltonian Cycle is NP-Hard. Conclusion: Since the Hamiltonian Cycle is both an NP-Problem and an NP-Hard. Therefore, it is an NP-complete problem.

- Peterson's graph is a great example of graphs where Hamiltonian Paths are present by Hamiltonian Cycle are not.
- See first item.

Problem 3: NP-Complete

Prove that the following problems are NP-complete.

1. Given two undirected graphs G and H, is G isomorphic to a subgraph of H?

   Solution:

   - Verification of feasibility: take both graphs and verify if for an edge in one, there is an equivalent edge in the other. It can be done in polynomial time. An easy alternative is to take edge by edge in both graph.

   - "Transformation": Reduction to Clique Problem.

   G and H are isomorphic if there is a structure that preserves a one-to-one correspondence between the vertices and edges.

2. Given an undirected graph G, does G have a spanning tree in which every node has degree at most 17?

   Solution:

   We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph G, let H be the graph obtained by attaching a fan of 15 edges to every vertex of G. Call a spanning tree of H almost-Hamiltonian if it has maximum degree 17. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

   - Suppose G has a Hamiltonian path P. Let T be the spanning tree of H obtained by adding every fan edge in H to P. Every vertex v of H is either a leaf of T or a vertex of P. If $v \in P$, then $deg_P(v) \leq 2$, and therefore $deg_H(v) = deg_P(v) + 15 \leq 17$. We conclude that H is an almost-Hamiltonian spanning tree.
   - Suppose H has an almost-Hamiltonian spanning tree T. The leaves of T are precisely the vertices of H with degree 1; these are also precisely the vertices of H that are not vertices of G. Let P be the subtree of T obtained by deleting every leaf of T. Observe that P is a spanning tree of G, and for every vertex $v \in P$, we have $deg_P(v) = deg_T(v) - 15 \leq 2$. We conclude that P is a Hamiltonian path in G.

   Given G, we can easily build H in polynomial time by brute force.

3. Given an undirected graph G, does G have a spanning tree with at most 42 leaves?

Solution:

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.1 Given an arbitrary graph G, let H be the graph obtained from G by adding the following vertices and edges:

- First we add a vertex z with edges to every other vertex in z.
- Then we add 41 vertices each with edges to t and nothing else.

Call a spanning tree of H almost-Hamiltonian if it has at most 42 leaves. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

- Suppose G has a Hamiltonian path P. Suppose P starts at vertex s and ends at vertex t. Let T be subgraph of H obtained by adding the edge tz and all possible edges l_i. Then T is a spanning tree of H with exactly 42 leaves, namely s and all 41 new vertices l_i.
- Suppose H has an almost-Hamiltonian spanning tree T. Every node l_i is a leaf of T, so T must consist of the 42 edges z'i and a simple path from z to some vertex s of G. Let t be the only neighbor of z in T that is not a leaf l_i, and let P be the unique path in T from s to t. This path visits every vertex of G; in other words, P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force.