Aalto University

Lecture Notes - Week IX

Polyhedral Theory

Fernando Dias, Philine Schiewe and Piyalee Pattanaik

March 11, 2024

CHAPTER 1
Linear Programming

So far in this course, we are solving either **algorithms** or an **integer linear programming formulation**.

In terms of ILP, they always follows the same format. Consider a model in the **general (or standard) form**:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.: } & g_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, l \\ & x \in X, \end{aligned}$$

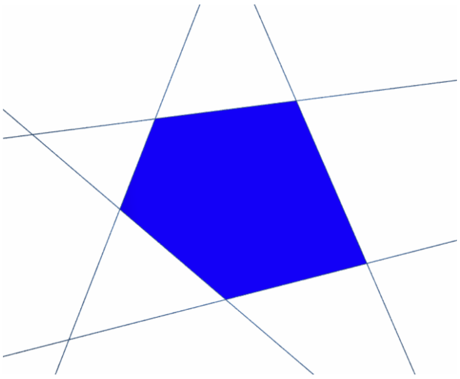
which is equivalent to a matrix format:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.:} & \\ & Ax \leq b \\ & -Ax \leq -b \\ & x \leq 0 \end{aligned}$$

or the brand-new polyhedral form:

$$\begin{aligned} \max \quad & c^T x^+ - c^T x^- \\ \text{s.t.:} & \\ & Ax^+ + Ax^- + Is = b \\ & x^+, x^-, s \geq 0 \\ & x = x^+ - x^- \end{aligned}$$

Based on those polytope, we can have different levels and dimensions such as An example of 2-D polytope:



and an example of 3-D polytope:

CHAPTER 2
Farkas Theorem

Farkas' theorem plays a central role in deriving optimality conditions. It can assume several alternative forms, typically referred to as Farkas' lemmas. In essence, Farkas' theorem demonstrates that a given system of linear equations has a solution if and only if a related system can be shown to have no solutions and vice-versa.

Theorem 1 *Let A be an $m \times n$ matrix and c be an n vector. Then exactly one of the following two systems has a solution:*

$$(1) : Ax \leq 0,\ c^\top x > 0,\ x \in \mathbb{R}^n$$
$$(2) : A^\top y = c,\ y \geq 0,\ y \in \mathbb{R}^m.$$

Proof 1 *Suppose (2) has a solution. Let x be such that $Ax \leq 0$. Then $c^\top x = (A^\top y)^\top x = y^\top Ax \leq 0$. Hence, (1) has no solution.*

Next, suppose (2) has no solution. Let $S = \{x \in \mathbb{R}^n : x = A^\top y,\ y \geq 0\}$.Notice that S is closed and convex and that $c \notin S$.

There exists $p \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $p^\top c > \alpha$ and $p^\top x \leq \alpha$ for $x \in S$.

As $0 \in S$, $\alpha \geq 0$ and $p^\top c > 0$. Also, $\alpha \geq p^\top A^\top y = y^\top Ap$ for $y \geq 0$. This implies that $Ap \leq 0$, and thus p satisfies (1).

The first part of the proof shows that if we assume that system (2) has a solution, then $c^\top x > 0$ cannot hold for $y \geq 0$. The second part shows that c can be seen as a point not belonging to the closed convex set S for which there is a separation hyperplane and that the existence of such plane implies that system (1) must hold. The set S is closed and convex since it is a conic combination of rows a_i, for $i = 1, \ldots, m$. Using the $0 \in S$, one can show that $\alpha \geq 0$.

The last part uses the identity $p^\top A^\top = (Ap)^\top$ and the fact that $(Ap)^\top y = y^\top Ap$. Notice that, since y can be arbitrarily large and α is a constant, $y^\top Ap \leq \alpha$ can only hold if $y^\top Ap \leq 0$, requiring that $p \leq 0$ since $y \geq 0$ from the definition of S.

Farkas' theorem has an interesting geometrical interpretation from this proof, as illustrated in Figures 2.1. Consider the cone C formed by the rows of A

$$C = \{c \in \mathbb{R}^n : c_j = \sum_{i=1}^{m} a_{ij} y_i,\ j = 1, \ldots, n,\ y_i \geq 0,\ i = 1, \ldots, m\}$$

The **polar cone** of C, denoted C^0, is formed by all vectors having angles of 90° or more with vectors in C. That is,

$$C^0 = \{x : Ax \leq 0\}.$$

Notice that (1) has a solution if the intersection between the polar cone C^0 and the positive (H^+ as defined earlier) half-space $H^+ = \{x \in \mathbb{R}^n : c^\top x > 0\}$ is not empty. If (2) has a solution, as at the beginning of the proof, then $c \in C$ and the intersection $C^0 \cap H^+ = \emptyset$. Now, if (2) does not have a solution, that is, $c \notin C$, then one can see that $C^0 \cap H^+$ cannot be empty, meaning that (1) has a solution.

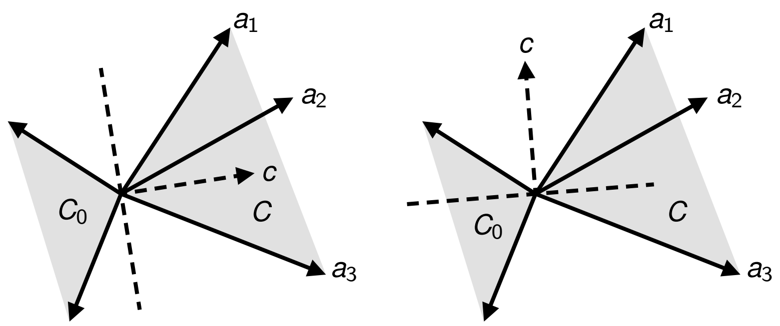


Figure 2.1: Geometrical illustration of the Farkas' theorem. On the left, system (2) has a solution, while on the right, system (1) has a solution