Lecture IX - Polyhedral Theory

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Previously on..

PREVIOUSLY ON...

- Dynamic Programming Theory;
- Knapsack Problem, Tower of Hanoi and Bellman-Ford algorithm;
- Enumeration;

Linear Programming

So far in this course, we are solving either **algorithms** or an **integer linear programming formulation**.

In terms of ILP, they always follows the same format. Consider a model in the **general (or standard) form**:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.:} \quad & g_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, l \\ & x \in X, \end{aligned}$$

Matrix Form

$$\max \quad c^T x$$

s.t.:

$$Ax \leq b$$

$$-Ax \leq -b$$

$$x \leq 0$$

Polyhedral Form

$$\max \quad c^T x^+ - c^T x^-$$

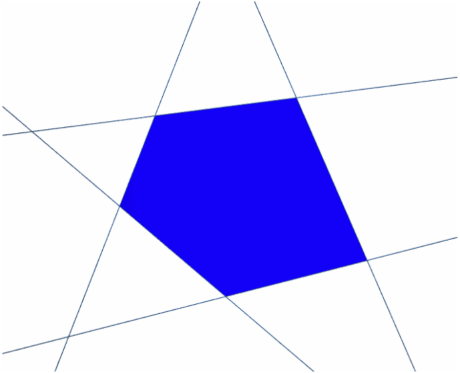
s.t.:

$$Ax^+ + Ax^- + Is = b$$

$$x^+, x^-, s \geq 0$$

$$x = x^+ - x^-$$

An example of 2-D polytope:



An example of 3-D polytope:

Farkas Theorem

Combinatorial Optimization
Previously on..
Linear Programming
Farkas Theorem
Farkas' theorem

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Theorem

Let A be an $m \times n$ matrix and c be an n-vector.
Then exactly one of the following two systems has a solution:

$$(1): \;\; Ax \leq 0, c^\top x > 0, x \in \mathcal{R}^n$$
$$(2): \;\; A^\top y = c, y \geq 0, y \in \mathcal{R}^m.$$

Farka's proof

Suppose (2) has a solution. Let x be such that $Ax \leq 0$. Then $c^\top x = (A^\top y)^\top x = y^\top Ax \leq 0$. Hence, (1) has no solution.

Next, suppose (2) has no solution. Let $S = \{x : x = A^\top y, y \geq 0\}$. Notice that S is closed and convex and that $c \notin S$. There exists $p \in \mathcal{R}^n$ and $\alpha \in \mathcal{R}$ such that $p^\top c > \alpha$ and $p^\top x \leq \alpha$ for $x \in S$.

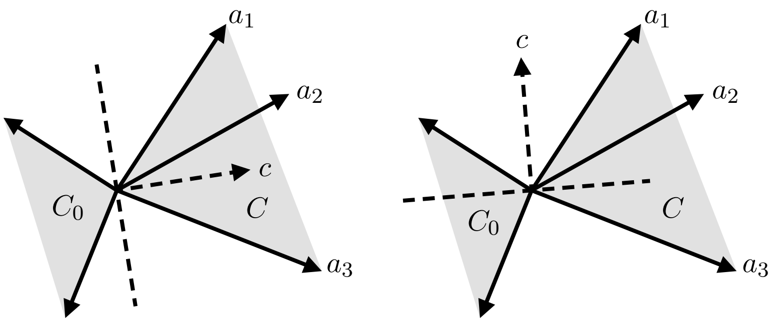
As $0 \in S$, $\alpha \geq 0$ and $p^\top c > 0$. Also, $\alpha \geq p^\top A^\top y = y^\top Ap$ for $y \geq 0$. This implies that $Ap \leq 0$, and thus p satisfies (1).

Geometry of the Farkas' theorem

Consider the cone formed by the rows a_i of A:
$C = \{c \in \mathcal{R}^n : c_j = \sum_{i=1}^{m} a_{ij} y_i, j = 1, \dots, n, y_i \geq 0, i = 1, \dots, m\}$. Its polar cone

is given by $C^0 = \{x : Ax \leq 0\}$. If $c \in C$, then (2) has a solution. Otherwise, (1) has a solution as $\{x : c^\top x > 0\} \cap C^0 \neq \emptyset$.

Geometry of the Farkas' theorem



Geometrical illustration of the Farkas' theorem. On the left, system (2) has a solution, while on the right, system (1) has a solution


thank you