Combinatorial Optimization

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Complexity - TSP

§ Week V §

Problem 1: TSP

Consider the Travelling Salesman Problem (TSP) where a salesman must visit each of n given cities $V = \{1, \cdots, n\}$ exactly once and then return to his starting point. The distance between two cities i and j is given by c_{ij}. The goal is to determine a tour of minimum length. We following ILP is proposed to solve this problem:

$$\textit{Minimize} \quad \sum_{i \in \{1,\ldots,n\}} \sum_{i \in \{1,\ldots,n\}} c_{ij} x_{ij} \tag{1.1a}$$

Subject to:

$$\sum_{j: j \neq i} x_{ij} = 1 \qquad \forall i \in \{1, \ldots, n\} \tag{1.1b}$$

$$\sum_{i: i \neq j} x_{ij} = 1 \qquad \forall j \in \{1, \ldots, n\} \tag{1.1c}$$

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} x_{ij} \leq |\mathcal{S}| - 1 \qquad \forall \emptyset \neq \mathcal{S} \subseteq \mathcal{V} \tag{1.1d}$$

$$x \in 0, 1 \tag{1.1e}$$

The goal is to show that this is a correct formulation of the TSP.

1. Give an interpretation of the binary variables x_{ij} and the constraints in the above program;
2. Prove that this formulation is correct. Start by showing that without the subtour elimination constraints every feasible solution to the above IP consists of vertex-disjoint cycles.
3. The following set of constraints are called the cut-set constraints for the TSP:

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{V} \setminus \mathcal{S}} x_{ij} \geq 1 \qquad \forall \emptyset \subseteq \mathcal{S} \subseteq \mathcal{V} \tag{1.2}$$

   Show that replacing the subtour elimination constraints by the cut-set constraints yields an alternative valid formulation of the TSP.

Problem 2: Graph Colouring

A k-colouring $k \in \mathcal{N}$ for an undirected graph $G = (V, E)$ is a surjective mapping $f : \mathcal{V} \rightarrow \{1, \cdots, k\}$ from the edges of G into the numbers $\{1, \cdots, k\}$ such that for each $(u, v) \in \mathcal{E}$ it holds $f(u) \neq f(v)$. The minimum graph colouring problem asks for the minimum k such that there exists a k-colouring in G. Give an IP-formulation for the minimum graph colouring problem and prove its correctness.

Problem 3: Linear vs Non-Linear

In contrast to non-linear programs, integer programs as discussed in this course cannot contain products of variables. In some cases, however, it is possible to replace such products in an integer program at the cost of introducing new variables. Consider the non-linear constraint $x \cdot y \leq b$ with $b \in \mathcal{R}_0^+$, where:

1. $x, y \in \{0, 1\}$;
2. $x \in [0, \mathcal{M}], \mathcal{M} \in \mathcal{R}_0^+, y \in \{0, 1\}$ (where M is a large constant);

and find an equivalent linear formulation for each.

Problem 4: Presidential Debate

Assume you are running for election in a country with five states $S = \{s_1, s_2, s_3, s_4, s_5\}$ and a two- party system. Each state $s \in S$ provides vs votes (where the sum $\sum_s v_s$ is odd) and you need at least half of these votes to win. To obtain all vs votes of the state s, you must win the popular vote in this state (or tie, tying is fine).

Currently, polls predict you will win a p_s fraction of the votes in state s while your political opponent is estimated to obtain a p_s fraction of them. Here $p_s \in [0, 1]$ and $\overline{p}_s \leq 1 - p_s$.

The election is imminent and your opponent is confident that they will win (which is an assumption made to justify them doing nothing to make the problem simpler). You, on the other hand, decide to use the remaining time to organise another eight campaign rallies. Because the election is right around the corner, you can only hold rallies in at most four of the five states and (strictly) more than three in the same state promise to be useless.

However, each of your rallies up to the third convinces a fraction f_s of the currently still undecided voters to vote for you. This means that your first rally in the state s would increase your percentage from p_s to $p_s + f_s(1 - p_s - \overline{p}_s)$. Note that this does have diminishing returns, the second rally is already only $(1 - f_s)$ as effective.

Use an IP to determine whether you can still win this election with the help of these rallies and where these would need to be. For the sake of this exercise, you may assume that the polls are entirely accurate (which is always the case, you have nothing to worry about).