Byt till polära koordinater i (x,y)-planet

$$\text{Volymen} = \iint_D z\,dx\,dy \overset{(z=y)}{=} \iint_D y\,dx\,dy =$$

$$= \int_0^{\pi/2}\int_0^a r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^a r^2 \sin\theta\,dr\,d\theta =$$

$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{r=0}^{r=a}\sin\theta\,d\theta = \int_0^{\pi/2}\frac{a^3}{3}\sin\theta\,d\theta$$

$$= \frac{a^3}{3}\left[-\cos\theta\right]_0^{\pi/2} = \frac{a^3}{3}\left(-0-(-1)\right) = \frac{a^3}{3}$$

Ex Beräkna $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

Vi kan inte skriva ner en primitiv funktion till e^{-x^2} bestående av elementära funktioner. Vi använder ett trick.

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)dy =$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = I^2$$

Vi beräknar I^2

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta =$$

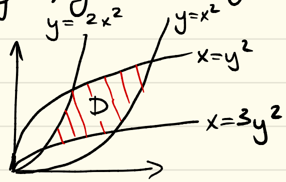
$$= 2\pi \lim_{N\to\infty} \int_0^N r e^{-r^2}\,dr = \begin{bmatrix} t = r^2 \\ dt = 2r\,dr \end{bmatrix} =$$

$$= \pi \lim_{N\to\infty} \int_0^{N^2} e^{-t}\,dt = \pi \lim_{N\to\infty} (1 - e^{-N^2}) = \pi$$

$$\Rightarrow \quad I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Ex Beräkna arean av området som begränsas av parablerna $y = x^2$, $y = 2x^2$, $x = y^2$ och $x = 3y^2$.

Lösning:



Inför $u = \dfrac{x^2}{y}$ och $v = \dfrac{y^2}{x}$

$$\frac{1}{2} \le u \le 1 \quad \text{och} \quad \frac{1}{3} \le v \le 1$$

$$\iint_D 1\,dx\,dy = \int_{1/3}^{1}\int_{1/2}^{1} 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial u}{\partial x} = \frac{2x}{y} \qquad \frac{\partial u}{\partial y} = -\frac{x^2}{y^2}$$

$$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2} \qquad \frac{\partial v}{\partial y} = \frac{2y}{x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4-1 = 3$$

$$\Longrightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$$

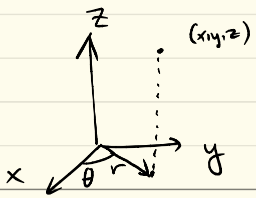
$$\text{Arean} = \iint_D 1\,dx\,dy = \int_{1/3}^{1}\int_{1/2}^{1} \frac{1}{3}\,du\,dv = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{9} \text{ a.e.}$$

I trippelintegraler (och multipelintegraler i allmänhet) funkar det likadant.

$$\begin{cases} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{cases} \qquad dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

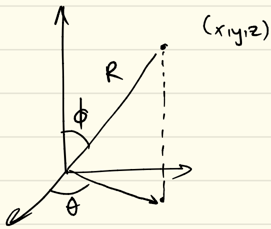
Cylindriska koordinater

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

Sfäriska koordinater

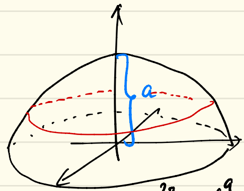


$$\begin{cases} x = R\sin\phi\cos\theta \\ y = R\sin\phi\sin\theta \\ z = R\cos\phi \end{cases}$$

$R > 0,\ \theta \in [0, 2\pi),\ \phi \in [0, \pi)$

$$dx\,dy\,dz = R^2 \sin\phi \, dR\, d\phi\, d\theta$$

Ex Ett halvklot H med radie a har densitet ρ som beror på avståndet R från centrum i basdisken enligt $\rho = k(2a - R)$. Beräkna halvklotets massa.



$$\text{Massan} = \iiint_H \rho(x,y,z)\,dx\,dy\,dz \underset{\substack{\uparrow \\ \text{Sfäriska} \\ \text{koordinater}}}{=} \int_0^{\frac{\pi}{2}} d\phi \int_0^{2\pi} d\theta \int_0^a k(2a - R)R^2 \sin\phi\, dR$$

$$= k \int_0^{\pi/2} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^a 2aR^2 - R^3\, dR =$$

$$= 2\pi k \int_0^{\pi/2} \sin\phi \left[\frac{2aR^3}{3} - \frac{R^4}{4} \right]_0^a = 2\pi k \left(\frac{8-3}{12} \right) a^4 =$$

$$= \frac{5\pi k a^4}{6}$$

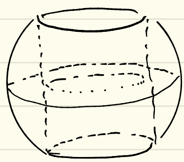
⊗

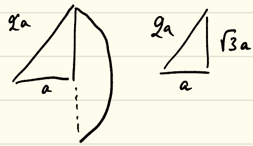
Tröghetsmoment (runt z-axeln) hos en kropp K ges av integralen $\iiint_K (x^2+y^2)\,\rho(x,y,z)\,dV$

där $\rho(x,y,z)$ är kroppens densitet.

<u>Ex</u> Beräkna tröghetsmomentet kring z-axeln för kroppen, med konstant densitet ρ, givet av olikheterna $x^2+y^2+z^2 \leq 4a^2$ och $x^2+y^2 \geq a^2$.

<u>Lösning:</u>





$$\iiint_K (x^2+y^2)\rho\,dx\,dy\,dz \overset{\text{Cyl. koord.}}{=} \rho \int_0^{2\pi} d\theta \int_a^{2a} r\,dr \int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r^2\,dz =$$

$$= \rho\, 2\pi \int_a^{2a} 2r^3\sqrt{4a^2-r^2}\,dr = \begin{bmatrix} t = 4a^2-r^2 & r^2 = 4a^2-t \\ dt = -2r\,dr & t_{2a}=0 \quad t_a = 3a^2 \end{bmatrix}$$

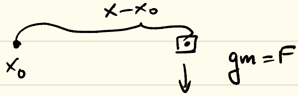
$$= -2\pi\rho \int_{3a^2}^{0} (4a^2-t)\,t^{1/2}\,dt = 2\pi\rho \int_0^{3a^2} 4a^2 t^{1/2} - t^{3/2}\,dt =$$

$$= 2\pi\rho \left[4a^2 \frac{t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2} \right]_0^{3a^2} =$$

$$= 2\pi\rho \left(\frac{8\cdot 3\sqrt{3}}{3} - \frac{18\sqrt{3}}{5} \right) a^5 = \frac{44}{5}\pi\rho a^5\sqrt{3}$$

Masscentrum

Den punkt (x_0, y_0, z_0) såm att vridmomenten är noll för punkten K.



$$\begin{cases} \iiint\limits_K (x-x_0)\,\rho(x,y,z)\,dV = 0 \\ \iiint\limits_K (y-y_0)\,\rho(x,y,z)\,dV = 0 \\ \iiint\limits_K (z-z_0)\,\rho(x,y,z)\,dV = 0 \end{cases}$$

Låt oss lösa detta för x_0

$$\iiint\limits_K x\,\rho(x,y,z)\,dV = \iiint\limits_K x_0\,\rho(x,y,z)\,dV$$

$$\Longrightarrow \quad x_0 = \frac{\iiint\limits_K x\,\rho(x,y,z)\,dV}{\iiint\limits_K \rho(x,y,z)\,dV}$$

På samma sätt

$$y_0 = \frac{\iiint\limits_K y\,\rho(x,y,z)\,dV}{\iiint\limits_K \rho(x,y,z)\,dV} \quad ; \quad z_0 = \frac{\iiint\limits_K z\,\rho(x,y,z)\,dV}{\iiint\limits_K \rho(x,y,z)\,dV}$$

Ex Beräkna masscentrum för K som ges av $x \geq 0, y \geq 0, z \geq 0$ och $x^2+y^2+z^2 \leq a^2$ $(\rho = 1)$

$$\iiint_K 1 \, dV = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^a R^2 \sin\phi \, dR =$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin\phi \, d\phi \left[\frac{R^3}{3} \right]_0^a = \frac{a^3 \pi}{6}.$$

$$\iiint_K x \, dV = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^a R^2 \sin\phi \, (R \sin\phi \cos\theta) \, dR =$$

$$= \int_0^{\pi/2} \sin^2\phi \, d\phi \int_0^{\pi/2} \cos\theta \, d\theta \int_0^a R^3 \, dR =$$

$$= \frac{a^4}{4} \int_0^{\pi/2} \sin^2\phi \, d\phi = \frac{a^4}{4} \int_0^{\pi/2} \frac{1 - \cos 2\phi}{2} \, d\phi =$$

$$= \frac{a^4}{4} \cdot \frac{\pi}{4} = \frac{\pi a^4}{16}$$

$$x_0 = \frac{\pi a^4}{16} \cdot \frac{6}{\pi a^3} = \frac{3a}{8}$$

$$\Longrightarrow \quad (x_0, y_0, z_0) = \left(\frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8} \right)$$