Aalto University

Discrete-time optimal control

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Overview

To formulate a general **discrete-time optimal control** problem, we combine the notions on dynamic systems and simulation with the notions on nonlinear programming

- We understand/treat general (discrete-time) optimal control problems as a special form of nonlinear programming and discuss its numerical solution

Overview (cont.)

Consider a system f which maps an initial state vector x_k onto a final state vector x_{k+1}

- We also consider the presence of a control u_k that affects the transition

$$x_{k+1} = f\left(x_k, u_k | \theta_x\right), \quad (k = 0, 1, \dots, K-1)$$

We consider transitions over an arbitrary time-horizon, from time $k = 0$ to time $k = K$

$$0 \cdots 1 \cdots \cdots (k-1) \cdots k \cdots (k+1) \cdots \cdots (K-1) \cdots K$$

Over said time-horizon, we have the following sequences of state and control variables

- ⇝ For the controls, we have $\{u_k\}_{k=0}^{K-1}$ with $u_k \in \mathcal{R}^{N_u}$
- ⇝ For the states, we have $\{x_k\}_{k=0}^{K}$ with $x_k \in \mathcal{R}^{N_x}$

For notational simplicity, we used time-invariant dynamics f

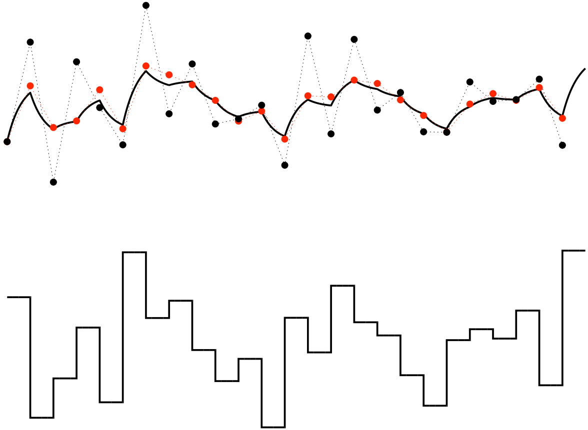
- In general, we may have $x_{k+1} = f_k\left(x_k, u_k | \theta_x\right)$

Overview (cont.)

$$x_{k+1} = f\left(x_k, u_k | \theta_x\right), \quad (k = 0, 1, \ldots, K-1)$$

The dynamics f are often derived from the discretisation of a continuous-time system

- As result of a numerical integration schemes, under piecewise constant controls



Overview (cont.)

$$x_{k+1} = f\left(x_k, u_k | \theta_x\right), \quad (k = 0, 1, \dots, K-1)$$

Given an initial state x_0 and some sequence of controls $\{u_k\}_{k=0}^{K-1}$, we know $\{x_k\}_{k=0}^{K}$

The forward-simulation function determines the sequence $\{x_k\}_{k=0}^{K}$ of visited states

$$\begin{aligned} f_{\text{sim}} &: \mathcal{R}^{N_x + (K \times N_u)} \rightarrow \mathcal{R}^{(K+1)N_x} \\ &: (x_0, u_0, u_1, \dots, u_{K-1}) \mapsto (x_0, x_1, \dots, x_K) \end{aligned}$$

For systems with no special structure, the forward-simulation map is built recursively

$$\begin{aligned} x_0 &= x_0 \\ x_1 &= f\left(x_0, u_0 | \theta_x\right) \\ x_2 &= f\left(x_1, u_1 | \theta_x\right) \\ &= f\left(f\left(x_0, u_0 | \theta_x\right), u_1 | \theta_x\right) \\ x_3 &= f\left(x_2, u_2 | \theta_x\right) \\ &= f\left(f\left(f\left(x_0, u_0 | \theta_x\right), u_1 | \theta_x\right), u_2 | \theta_x\right) \\ \cdots &= \cdots \end{aligned}$$

Overview (cont.)

$$x_{k+1} = f\left(x_k, u_k | \theta_x\right), \quad (k = 0, 1, \dots, K-1)$$

In optimal control, the dynamics can be used as equality constraints in optimisation

In this case, the initial state vector x_0 is not necessarily known, nor its is fixed

- Therefore, it can be one of the decision variables to be determined
- Moreover, certain additional constraints may be required to it

Similarly, also the final state x_K can be treated as decision variable in an optimisation

Overview (cont.)

Initial and terminal state constraints

We express the constraints on initial and terminal states in terms of function $r(x_0, x_K)$

$$r : \mathcal{R}^{N_x + N_x} \rightarrow \mathcal{R}^{N_r}$$

We express the desire to reach certain initial and terminal states as equality constraints

$$r(x_0, x_K) = 0$$

For fixed initial state $x_0 = \overline{x}_0$, we have

$$r(x_0, x_K) = x_0 - \bar{x}_0$$

For fixed terminal state $x_K = \overline{x}_K$, we have

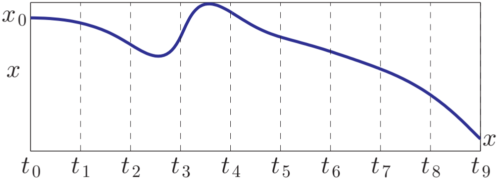
$$r(x_0, x_K) = x_K - \bar{x}_K$$

For fixed both initial and terminal states, $x_0 = \overline{x}_0$ and $x_K = \overline{x}_K$, we have

$$r(x_0, x_K) = \begin{bmatrix} x_0 - \bar{x}_0 \\ x_K - \bar{x}_K \end{bmatrix}$$

Overview (cont.)

When both the initial and terminal states are fixed ($x_0 = \overline{x}_0$ and $x_K = \overline{x}_K$), we have



$$r\left(x_0, x_K\right) = \underbrace{\begin{bmatrix} x_0^{(1)} - \overline{x}_0^{(1)} \\ x_0^{(2)} - \overline{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \overline{x}_0^{(N_x)} \\ \hline x_K^{(1)} - \overline{x}_K^{(1)} \\ x_K^{(2)} - \overline{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \overline{x}_K^{(N_x)} \end{bmatrix}}_{N_r \times 1}$$

Overview (cont.)

Path constraints

We express certain constraints on state and control values x_k and u_k along their path

- ⇝ These constraints often represent technological restrictions and/or desiderata
- ⇝ They are commonly expressed in terms of inequality constraints
- The main idea is to use them to prevent operational violations

$$h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1$$

For notational simplicity, we used time-invariant inequality constraint functions h

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For common upper and lower bounds on the controls, $u_{\min} \leq u_k \leq u_{\max}$, we have

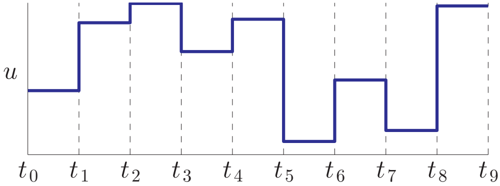
$$h(x_k, u_k) = \begin{bmatrix} u_k - u_{\max} \\ u_{\min} - u_k \end{bmatrix}$$

For common upper and lower bounds on the states, $x_{\min} \leq x_k \leq x_{\max}$, we have

$$h(x_k, u_k) = \begin{bmatrix} x_k - x_{\max} \\ x_{\min} - x_k \end{bmatrix}$$

Overview (cont.)

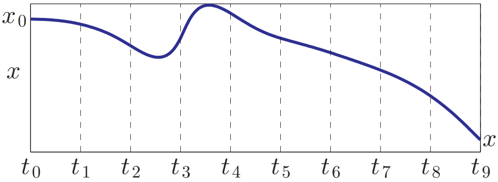
For common upper and lower bounds on the controls, $u_{\min} \geq u_k \geq u_{\max}$, we have



$$h\left(x_k, u_k\right) = \left[\begin{array}{c} u_k^{(1)} - u_{\max}^{(1)} \\ u_k^{(2)} - u_{\max}^{(2)} \\ \vdots \\ u_k^{(N_u)} - u_{\max}^{(N_u)} \\ \hline u_{\min}^{(1)} - u_k^{(1)} \\ u_{\min}^{(2)} - u_k^{(2)} \\ \vdots \\ u_{\min}^{(N_u)} - u_k^{(N_u)} \end{array}\right]$$

Overview (cont.)

For common upper and lower bounds on the states, $x_{\min} \geq x_k \geq x_{\max}$, we have



$$h\left(x_k, u_k\right) = \begin{bmatrix} x_k^{(1)} - x_{\max}^{(1)} \\ x_k^{(2)} - x_{\max}^{(2)} \\ \vdots \\ x_k^{(N)} - x_{\max}^{(N_x)} \\ \hline x_{\min}^{(1)} - x_k^{(1)} \\ x_{\min}^{(2)} - x_k^{(2)} \\ \vdots \\ x_{\min}^{(N_x)} - x_k^{(N_x)} \end{bmatrix}$$

Problem formulations

Discrete-time optimal control

Problem formulations

We are given system dynamics and specifications on the state and control constraints

We use them to formulate the discrete-time optimal control problem

- It is a general constrained nonlinear program

$$\underbrace{\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}}}_{\text{Decision variables}} \quad \underbrace{E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)}_{\text{Objective function}}$$

$$\text{subject to} \quad \underbrace{x_{k+1} - f(x_k, u_k | \theta_x) = 0}_{\text{Equality constraints}}, \quad k = 0, 1, \dots, K-1$$

$$\underbrace{h(x_k, u_k) \leq 0}_{\text{Inequality constraints}}, \quad k = 0, 1, \dots, K-1$$

$$\underbrace{r(x_0, x_K) = 0}_{\text{Equality constraints}}$$

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad && k = 0, 1, \dots, K-1 \\
& h(x_k, u_k) \leq 0, && k = 0, 1, \dots, K-1 \\
& r(x_0, x_K) = 0
\end{aligned}$$

The **objective function**, in general two terms

$$\sum_{k=0}^{K-1} L(x_k, u_k) + E(x_K)$$

The **decision variables**, in general two sets

$$x_0, x_1, \dots, x_{K-1}, x_K$$
$$u_0, u_1, \dots, u_{K-1}$$

The **equality constraints**, in general two sets

$$x_{k+1} - f(x_k, u_k | \theta_x) = 0 \quad (k = 0, \dots, K-1)$$
$$r(x_0, x_K) = 0$$

The **inequality constraints**

$$h(x_k, u_k) \leq 0 \quad (k = 0, 1, \dots, K-1)$$

Problem formulations (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \qquad k = 0, 1, \dots, K-1 \\
& h(x_k, u_k) \leq 0, \qquad k = 0, 1, \dots, K-1 \\
& r(x_0, x_K) = 0
\end{aligned}$$

The **objective function** is the sum of all **stage costs** $L(x_k, u_k)$ and a **terminal cost** $E(x_K)$

$$\underbrace{\sum_{k=0}^{K-1} L(x_k, u_k) + E(x_K)}_{f(w) \in \mathcal{R}}$$

That is,

$$L(x_0, u_0) + L(x_1, u_1) + \cdots + L(x_{K-1}, u_{K-1}) + E(x_K)$$

Stage cost is a (potentially nonlinear and time-varying) function of state and controls

The terminal cost is a (potentially nonlinear and time-varying) function of state

Problem formulations (cont.)

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \qquad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \qquad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

The **decision variables** are both the $K \times N_u$ **controls** and the $(K+1) \times N_x$ **state variables**

$$\underbrace{\underbrace{(x_0, x_1, \dots, x_{K-1}, x_K)}_{\text{State variables}} \cup \underbrace{(u_0, u_1, \dots, u_{K-1})}_{\text{Control variables}}}_{w \in \mathcal{R}^{K \times N_u + (K+1) \times N_x}}$$

Problem formulations (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \qquad k = 0, 1, \dots, K-1 \\
& h(x_k, u_k) \leq 0, \qquad k = 0, 1, \dots, K-1 \\
& r(x_0, x_K) = 0
\end{aligned}$$

The **equality constraints** consist of the K **dynamics** and the N_r **boundary conditions**

$$\underbrace{\begin{matrix} x_{k+1} - f(x_k, u_k | \theta_x) = 0 \quad (k = 0, \dots, K-1) \\ r(x_0, x_K) = 0 \end{matrix}}_{g(w) \in \mathcal{R}^{N_g}}$$

The **inequality constraints**

$$\underbrace{h(x_k, u_k) \leq 0 \quad (k = 0, 1, \dots, K-1)}_{h(w) \in \mathcal{R}^{N_h}}$$

Problem formulations (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \qquad k = 0, 1, \dots, K-1 \\
& h(x_k, u_k) \leq 0, \qquad k = 0, 1, \dots, K-1 \\
& r(x_0, x_K) = 0
\end{aligned}$$

The discrete-time optimal control problem is a potentially very large nonlinear program

- In principle, its solution can be approached using any generic NLP solver

We introduce the two approaches used to solve discrete-time optimal control problems

⇝ The **simultaneous approach**

⇝ The **sequential approach**

The simultaneous approach

Problem formulations

Problem formulations | Simultaneous approach

$$\begin{aligned}
\min_{\substack{x_0, x_1, \ldots, x_K \\ u_0, u_1, \ldots, u_{K-1}}} \quad & E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right) \\
\text{subject to} \quad & x_{k+1} - f\left(x_k, u_k | \theta_x\right) = 0, \quad && k = 0, 1, \ldots, K-1 \\
& h\left(x_k, u_k\right) \leq 0, && k = 0, 1, \ldots, K-1 \\
& r\left(x_0, x_K\right) = 0
\end{aligned}$$

The **simultaneous approach** solves the problem in the space of all the decision variables

$$w = \left(x_0, u_0, x_1, u_1, \ldots, x_{K-1}, u_{K-1}, x_K\right)$$

Thus, there are $(K \times N_u) + ((K+1) \times N_x)$ decision variables

Problem formulations | Simultaneous approach

The Lagrangian function of the problem,

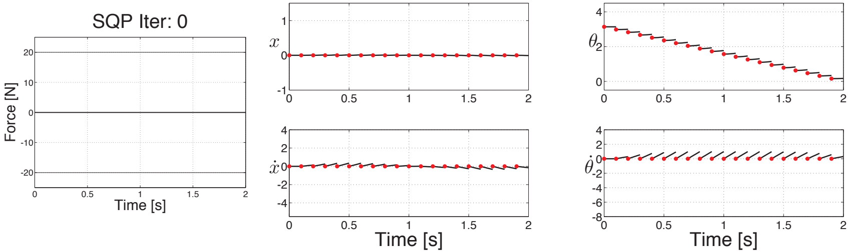
$$\mathcal{L}(w, \lambda, \mu) = f(w) + \lambda^T g(w) + \mu^T h(w)$$

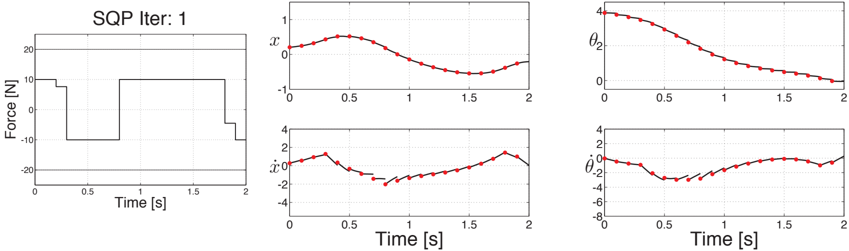
The Karush-Kuhn-Tucker conditions,

$$\begin{aligned} \nabla f(w^*) + \nabla g(w^*)\lambda^* + \nabla h(w^*)\mu^* &= 0 \\ g(w^*) &= 0 \\ h(w^*) &\leq 0 \\ \mu^* &\geq 0 \\ \mu^*_{n_h} h_{n_h}(w^*) &= 0, \quad n_h = 1, \dots, N_h \end{aligned}$$

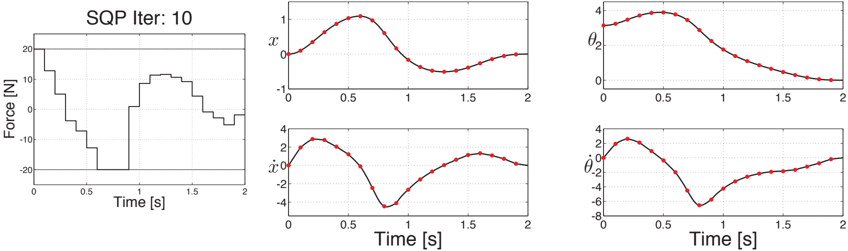
If point $w^* = (x_0^*, u_0^*, \dots, x_{K-1}^*, u_{K-1}^*, x_K^*)$ is a local minimiser of the nonlinear program and if LICQ holds at w^*, there there exist two vectors, the Lagrange multipliers $\lambda \in \mathcal{R}^{N_g}$ and $\mu \in \mathcal{R}^{N_h}$, such that the Karhush-Kuhn-Tucker conditions are verified

Problem formulations | Simultaneous approach (cont.)





Problem formulations | Simultaneous approach (cont.)



Problem formulations | Simultaneous approach (cont.)

To understand more closely the structure and sparsity properties, consider an example

$$\begin{aligned}\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0\end{aligned}$$

We consider a discrete-time optimal control problem with no inequality constraints

- (The inequality constraints are omitted for notational simplicity)

The objective function $\underbrace{f(w) = E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)}$ of the decision variables

$$w = \left(\underbrace{x_0, u_0}, \underbrace{x_1, u_1}, \dots, \underbrace{x_{K-1}, u_{K-1}}, \underbrace{x_K}\right)$$

Problem formulations | Simultaneous approach (cont.)

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & \underbrace{x_{k+1} - f(x_k, u_k | \theta_x) = 0}, \quad k = 0, 1, \dots, K-1 \\ & \underbrace{r(x_0, x_K) = 0} \end{aligned}$$

We define the equality constraint function $g((w)$ by joining all the equality constraints

$$g(w) = \begin{bmatrix} g_1(w) \\ g_2(w) \\ \vdots \\ g_{N_g}(w) \end{bmatrix} = \underbrace{\left[\begin{array}{c} x_1 - f(x_0, u_0) \\ x_2 - f(x_1, u_1) \\ \vdots \\ x_K - f(x_{K-1}, u_{K-1}) \\ \hline r(x_0, x_K) \end{array}\right]}_{((K \times N_x) + N_r) \times 1}$$

Problem formulations | Simultaneous approach (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\
& r(x_0, x_K) = 0
\end{aligned}$$

We define the Lagrangian function (objective function and equality constraints, $\mathcal{L}(w, \lambda)$)

$$\mathcal{L}(w, \lambda) = f(w) + \lambda^T g(w)$$

The $N_g = (K \times N_x) + N_r$ equality multipliers can be any real numbers λ_{n_g}

$$\lambda = \Big(\underbrace{\lambda_1, \lambda_2, \dots, \underbrace{\lambda_k}, \dots, \lambda_K,}_{\text{Dynamics}} \underbrace{\lambda_{N_r}}_{\text{Boundaries}} \Big)$$

First-order optimality is given by the KKT conditions

$$\begin{aligned}
\nabla_w \mathcal{L}(w, \lambda) &= 0 \\
g(w) &= 0
\end{aligned}$$

Problem formulations | Simultaneous approach (cont.)

$$\underbrace{\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_k & \cdots & \lambda_{K-1} & \lambda_K & | & \lambda_{N_r}\end{bmatrix}}_{\lambda^T}\underbrace{\begin{bmatrix} x_1 - f(x_0, u_0) \\ x_2 - f(x_1, u_1) \\ \vdots \\ x_k - f(x_{k-1}, u_{k-1}) \\ \vdots \\ x_{K-1} - f(x_{K-2}, u_{K-2}) \\ x_K - f(x_{K-1}, u_{K-1}) \\ \hline r(x_0, x_K)\end{bmatrix}}_{g(w)}$$

After expanding the terms in the inner product, we re-write the Lagrangian function

$$\mathcal{L}(w, \lambda) = \underbrace{E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)}_{f(w)} + \underbrace{\left(\sum_{k=0}^{K-1} \lambda_{k+1}^T \left(f(x_k, u_k) - x_{k+1}\right) + \lambda_{N_r}^T r(x_0, x_K)\right)}_{\lambda^T g(w)}$$

Problem formulations | Simultaneous approach (cont.)

Consider one, at any time $k = 1, 2, \ldots, K-1$, of the dynamic (equality) constraints

$$x_{k+1} - f\left(x_k, u_k\right) = 0$$

After expanding these equality constraints, more explicitly we have

$$\underbrace{\begin{bmatrix} x_{k+1}^{(1)} - f_1\left(x_k, u_k\right) \\ x_{k+1}^{(2)} - f_2\left(x_k, u_k\right) \\ \vdots \\ x_{k+1}^{(n_x)} - f_{n_x}\left(x_k, u_k\right) \\ \vdots \\ x_{k+1}^{(N_x-1)} - f_{N_x-1}\left(x_k, u_k\right) \\ x_{k+1}^{(N_x)} - f_{N_x}\left(x_k, u_k\right) \end{bmatrix}}_{N_x \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Problem formulations | Simultaneous approach (cont.)

Consider the associated inner product with the corresponding equality multiplier,

$$\underbrace{\underbrace{\lambda_{k+1}^T}_{1\times N_x}\underbrace{\left(f\left(x_k, u_k\right)-x_{k+1}\right)}_{N_x\times 1}}_{1\times 1}$$

After expanding the inner product, more explicitly we have

$$\underbrace{\begin{bmatrix}\lambda_{k+1}^{(1)} & \lambda_{k+1}^{(2)} & \cdots & \lambda_{k+1}^{(n_x)} & \cdots & \lambda_{k+1}^{(N_x-1)} & \lambda_{k+1}^{(N_x)}\end{bmatrix}}_{1\times N_x}\underbrace{\begin{bmatrix} x_{k+1}^{(1)}-f_1\left(x_k, u_k\right) \\ x_{k+1}^{(2)}-f_2\left(x_k, u_k\right) \\ \vdots \\ x_{k+1}^{(n_x)}-f_{n_x}\left(x_k, u_k\right) \\ \vdots \\ x_{k+1}^{(N_x-1)}-f_{N_x-1}\left(x_k, u_k\right) \\ x_{k+1}^{(N_x)}-f_{N_x}\left(x_k, u_k\right)\end{bmatrix}}_{N_x\times 1}$$

Problem formulations | Simultaneous approach (cont.)

Similarly, consider the boundary constraint on the initial ad terminal state

$$r\left(x_0, x_K\right) = 0$$

After expanding also these equality constraints, more explicitly we have

$$r\left(x_0, x_N\right) = \underbrace{\begin{bmatrix} x_0^{(1)} - \overline{x}_0^{(1)} \\ x_0^{(2)} - \overline{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \overline{x}_0^{(N_x)} \\ \hline x_K^{(1)} - \overline{x}_K^{(1)} \\ x_K^{(2)} - \overline{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \overline{x}_K^{(N_x)} \end{bmatrix}}_{N_r \times 1}$$

Problem formulations | Simultaneous approach (cont.)

Consider the inner product $\lambda_{N_r}^T r(x_0, x_K)$ with the corresponding equality multiplier,

$$\underbrace{\underbrace{\lambda_{N_r}^T}_{1 \times N_r} \underbrace{r(x_0, x_K)}_{N_r \times 1}}_{1 \times 1}$$

After expanding the inner product, we have

$$\underbrace{\begin{bmatrix} \lambda_{N_r}^{(1)} & \lambda_{N_r}^{(1)} & \cdots & \lambda_{N_r}^{(N_x)} & | & \lambda_{N_r}^{(N_x+1)} & \lambda_{N_r}^{(N_x+2)} & \cdots & \lambda_{N_r}^{(2N_x)} \end{bmatrix}}_{1 \times N_r} \underbrace{\begin{bmatrix} x_0^{(1)} - \overline{x}_0^{(1)} \\ x_0^{(2)} - \overline{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \overline{x}_0^{(N_x)} \\ \hline x_K^{(1)} - \overline{x}_K^{(1)} \\ x_K^{(2)} - \overline{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \overline{x}_K^{(N_x)} \end{bmatrix}}_{N_r \times 1}$$

Problem formulations | Simultaneous approach (cont.)

Putting things together, the Lagrangian function for equality constrained problems

$$\mathcal{L}(w,\lambda) = \underbrace{f(w)}_{1\times 1} + \underbrace{\begin{bmatrix} \underbrace{\lambda_1}_{1\times N_x} & \underbrace{\lambda_2}_{1\times N_x} & \cdots & \underbrace{\lambda_K}_{1\times N_x} & \underbrace{\lambda_{N_r}}_{1\times N_r} \end{bmatrix}}_{\underbrace{\lambda^T}_{1\times((K\times N_x)+N_r)}} \underbrace{\left[\begin{array}{c} \underbrace{x_1 - f(x_0, u_0)}_{N_x\times 1} \\ \underbrace{x_2 - f(x_1, u_1)}_{N_x\times 1} \\ \vdots \\ \underbrace{x_K - f(x_{K-1}, u_{K-1})}_{N_x\times 1} \\ \hline \underbrace{r(x_0, x_K)}_{N_r\times 1} \end{array}\right]}_{\underbrace{g(w)}_{((K\times N_x)+N_r)\times 1}}$$

Problem formulations | Simultaneous approach (cont.)

$$
\begin{aligned}
\nabla_w \mathcal{L}(w, \lambda) &= 0 \\
g(w) &= 0
\end{aligned}
$$

The first KKT condition regards the derivative of $\mathcal{L}$ with respect to the primal variables

$$w = (x_0, u_0, x_1, u_1, \dots, x_{K-1}, u_{K-1}, x_K)$$

The Lagrangian function $\mathcal{L}(w, \lambda)$ in structural (expanded) form,

$$\underbrace{\underbrace{E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)}_{f(w)} + \underbrace{\left(\sum_{k=0}^{K-1} \lambda_{k+1}^T \left(f(x_k, u_k) - x_{k+1}\right) + \lambda_{N_r}^T r(x_0, x_K)\right)}_{\lambda^T g(w)}}_{\mathcal{L}(w,\lambda)}$$

The second KKT condition collects all the equality constraints

$$
\begin{aligned}
x_{k+1} - f(x_k, u_k) &= 0 \quad (k = 0, \dots, K-1) \\
r(x_0, x_K) &= 0
\end{aligned}
$$

Problem formulations | Simultaneous approach (cont.)

$$g\left(w\right) = 0$$

For the second KKT condition, we have the equalities

$$\begin{aligned} x_{k+1} - f\left(x_k, u_k\right) &= 0 \quad (k = 0, \dots, K-1) \\ r\left(x_0, x_K\right) &= 0 \end{aligned}$$

That is, in a slightly more expanded form

$$\left[\begin{array}{c} \underbrace{x_1 - f\left(x_0, u_0\right)}_{N_x \times 1} \\ \underbrace{x_2 - f\left(x_1, u_1\right)}_{N_x \times 1} \\ \vdots \\ \underbrace{x_K - f\left(x_{K-1}, u_{K-1}\right)}_{N_x \times 1} \\ \hline \underbrace{r\left(x_0, x_K\right)}_{N_r \times 1} \end{array}\right] = \left[\begin{array}{c} \underbrace{0}_{N_x \times 1} \\ \underbrace{0}_{N_x \times 1} \\ \vdots \\ \underbrace{0}_{N_x \times 1} \\ \hline \underbrace{0}_{N_r \times 1} \end{array}\right]$$

Problem formulations | Simultaneous approach (cont.)

$$\nabla_w \mathcal{L}\left(w, \lambda\right) = 0$$

Consider the gradient of the Lagrangian function with respect to the primal variables

$$w = \left(x_0, u_0, x_1, u_1, \dots, x_{K-1}, u_{K-1}, x_K\right)$$

It is a concatenation of gradients of $\mathcal{L}(w, \lambda)$, each with respect to a primal variable

$$\nabla_w \mathcal{L}\left(w, \lambda\right) = \left[\begin{array}{c} \nabla_{x_0} \mathcal{L}\left(w, \lambda\right) \\ \nabla_{x_1} \mathcal{L}\left(w, \lambda\right) \\ \vdots \\ \nabla_{x_K} \mathcal{L}\left(w, \lambda\right) \\ \hline \nabla_{u_0} \mathcal{L}\left(w, \lambda\right) \\ \nabla_{u_1} \mathcal{L}\left(w, \lambda\right) \\ \vdots \\ \nabla_{u_{K-1}} \mathcal{L}\left(w, \lambda\right) \end{array}\right]$$

For the first KKT conditions, it is necessary to determine/evaluate derivatives

Problem formulations | Simultaneous approach (cont.)

$$\underbrace{E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right) + \left(\sum_{k=0}^{K-1} \lambda_{k+1}^T \left(f\left(x_k, u_k\right) - x_{k+1}\right) + \lambda_{N_r}^T r\left(x_0, x_K\right)\right)}_{\mathcal{L}(w,\lambda)}$$

Consider the derivatives of the Lagrangian function with respect to state variables x_k

- For $k = 0$, we have

$$\nabla_{x_0} \mathcal{L}\left(w, \lambda\right) = \nabla_{x_0} L\left(x_0, u_0\right) + \frac{\partial f\left(x_0, u_0\right)}{\partial x_0}^T \lambda_1 + \frac{\partial r\left(x_0, x_K\right)}{\partial x_0}^T \lambda_{N_r}$$

- For any $k = 1, \ldots, K-1$, we have

$$\nabla_{x_k} \mathcal{L}\left(w, \lambda\right) = \nabla_{x_k} L\left(x_k, u_k\right) + \frac{\partial f\left(x_k, u_k\right)}{\partial x_k}^T \lambda_{k+1} - \lambda_k$$

- For $k = K$, we have

$$\nabla_{x_K} \mathcal{L}\left(w, \lambda\right) = \nabla_{x_K} E\left(x_K\right) - \lambda_K + \frac{\partial r\left(x_0, x_K\right)}{\partial x_K}^T \lambda_{N_r}$$

Problem formulations | Simultaneous approach (cont.)

Consider the generic term $\nabla_{x_k}\mathcal{L}(w,\lambda) = \underbrace{\nabla_{x_k}L(x_k,u_k)} + \dfrac{\partial f(x_k,u_k)}{\partial x_k}^T\lambda_{k+1} - \lambda_k$, at k

After expanding the first expression, we have

$$\nabla_{x_k}\mathcal{L}(w,\lambda) = \underbrace{\begin{bmatrix} \dfrac{\partial \mathcal{L}(w,\lambda)}{\partial x_k^{(1)}} \\ \dfrac{\partial \mathcal{L}(w,\lambda)}{\partial x_k^{(2)}} \\ \vdots \\ \dfrac{\partial \mathcal{L}(w,\lambda)}{\partial x_k^{(N_x)}} \end{bmatrix}}_{N_x \times 1}$$

Problem formulations | Simultaneous approach (cont.)

$$\nabla_{x_k} \mathcal{L}(w, \lambda) = \nabla_{x_k} L(x_k, u_k) + \underbrace{\frac{\partial f(x_k, u_k)}{\partial x_k}^T} \lambda_{k+1} - \lambda_k$$

Consider the derivative of the dynamics $f(x_k, u_k)$ with respect to state variables x_k,

$$\frac{\partial f(x_k, u_k)}{\partial x_k}$$

Remember that for the dynamics $f(x_k, u_k)$, we have the component functions

$$f(x_k, u_k) = \begin{bmatrix} f_1\left(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k\right) \\ \vdots \\ f_{n_x}\left(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k\right) \\ \vdots \\ f_{N_x}\left(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k\right) \end{bmatrix}$$

Problem formulations | Simultaneous approach (cont.)

$$f\left(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k\right) = \begin{bmatrix} f_1\left(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k\right) \\ \vdots \\ f_{n_x}\left(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k\right) \\ \vdots \\ f_{N_x}\left(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k\right) \end{bmatrix}$$

Thus, we have the corresponding component terms for the derivative of the dynamics

$$\frac{\partial f\left(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k\right)}{\partial x_k} = \begin{bmatrix} \dfrac{\partial f_1\left(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k\right)}{\partial x_k} \\ \vdots \\ \dfrac{\partial f_{n_x}\left(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k\right)}{\partial x_k} \\ \vdots \\ \dfrac{\partial f_{N_x}\left(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k\right)}{\partial x_k} \end{bmatrix}$$

Problem formulations | Simultaneous approach (cont.)

After further expanding the expression to highlight all of its terms, we have

$$\frac{\partial f\left(x_k, u_k\right)}{\partial x_k} = \underbrace{\begin{bmatrix} \dfrac{\partial f_1\left(x_k, u_k\right)}{\partial x_k^{(1)}} & \dfrac{\partial f_1\left(x_k, u_k\right)}{\partial x_k^{(2)}} & \cdots & \dfrac{\partial f_1\left(x_k, u_k\right)}{\partial x_k^{(N_x)}} \\ \dfrac{\partial f_2\left(x_k, u_k\right)}{\partial x_k^{(1)}} & \dfrac{\partial f_2\left(x_k, u_k\right)}{\partial x_k^{(2)}} & \cdots & \dfrac{\partial f_2\left(x_k, u_k\right)}{\partial x_k^{(N_x)}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_{N_x}\left(x_k, u_k\right)}{\partial x_k^{(1)}} & \dfrac{\partial f_{N_x}\left(x_k, u_k\right)}{\partial x_k^{(2)}} & \cdots & \dfrac{\partial f_{N_x}\left(x_k, u_k\right)}{\partial x_k^{(N_x)}} \end{bmatrix}}_{N_x \times N_x}$$

For the inner product with the associated equality multiplier, we get

$$\underbrace{\underbrace{\frac{\partial f\left(x_k, u_k\right)}{\partial x_k}^T}_{N_x \times N_x} \underbrace{\lambda_{k+1}}_{N_x \times 1}}_{N_x \times 1}$$

Problem formulations | Simultaneous approach (cont.)

$$\nabla_{x_0}\mathcal{L}(w,\lambda) = \nabla_{x_0}L(x_0,u_0) + \frac{\partial f(x_0,u_0)}{\partial x_0}^T\lambda_1 + \underbrace{\frac{\partial r(x_0,x_K)}{\partial x_0}^T}\lambda_{N_r}$$

Consider the derivatives of the boundary conditions with respect to x_0

$\rightsquigarrow$

$$\frac{\partial r(x_0,x_K)}{\partial x_0}$$

$$\nabla_{x_K}\mathcal{L}(w,\lambda) = \nabla_{x_K}E(x_K) - \lambda_K + \frac{\partial r(x_0,x_K)}{\partial x_K}^T\lambda_{N_r}$$

Consider the derivatives of the boundary conditions with respect to x_K

$\rightsquigarrow$

$$\frac{\partial r(x_0,x_K)}{\partial x_K}$$

Problem formulations | Simultaneous approach (cont.)

Remember that for the boundary constraints on the initial ad terminal state, we have

$$r\left(x_0, x_K\right) = \underbrace{\begin{bmatrix} x_0^{(1)} - \overline{x}_0^{(1)} \\ x_0^{(2)} - \overline{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \overline{x}_0^{(N_x)} \\ \hline x_K^{(1)} - \overline{x}_K^{(1)} \\ x_K^{(2)} - \overline{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \overline{x}_K^{(N_x)} \end{bmatrix}}_{N_r \times 1}$$

Problem formulations | Simultaneous approach (cont.)

For the derivative of the boundary constraints with respect to x_0, we have

$$\frac{\partial r\left(\underbrace{x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}}, x_K\right)}{\partial x_0} = \left[\begin{array}{c} \dfrac{\partial r_1\left(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K\right)}{\partial x_0} \\ \dfrac{\partial r_2\left(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K\right)}{\partial x_0} \\ \vdots \\ \dfrac{\partial r_{N_x}\left(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K\right)}{\partial x_0} \\ \hline \dfrac{\partial r_{N_x+1}\left(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K\right)}{\partial x_0} \\ \dfrac{\partial r_{N_x+2}\left(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K\right)}{\partial x_0} \\ \vdots \\ \dfrac{\partial r_{2N_x}\left(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K\right)}{\partial x_0} \end{array}\right]$$

Problem formulations | Simultaneous approach (cont.)

After further expanding the expression to highlight all of its terms, we have

$$\frac{\partial r\left(x_0, x_K\right)}{\partial x_0} = \underbrace{\begin{bmatrix} \dfrac{\partial r_1\left(x_0, x_K\right)}{\partial x_0^{(1)}} & \dfrac{\partial r_1\left(x_0, x_K\right)}{\partial x_0^{(2)}} & \cdots & \dfrac{\partial r_1\left(x_0, x_K\right)}{\partial x_0^{(N_x)}} \\ \dfrac{\partial r_2\left(x_0, x_K\right)}{\partial x_0^{(1)}} & \dfrac{\partial r_2\left(x_0, x_K\right)}{\partial x_0^{(2)}} & \cdots & \dfrac{\partial r_2\left(x_0, x_K\right)}{\partial x_0^{(N_x)}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial r_{2N_r}\left(x_0, x_k\right)}{\partial x_0^{(1)}} & \dfrac{\partial r_{2N_r}\left(x_0, x_K\right)}{\partial x_0^{(2)}} & \cdots & \dfrac{\partial r_{2N_r}\left(x_0, x_K\right)}{\partial x_0^{(N_x)}} \end{bmatrix}}_{2N_r \times N_x}$$

For the inner product with the associated equality multiplier, we get

$$\underbrace{\underbrace{\frac{\partial r\left(x_0, x_K\right)}{\partial x_0}^T}_{N_x \times 2N_r} \underbrace{\lambda_{N_r}}_{2N_r \times 1}}_{N_x \times 1}$$

Problem formulations | Simultaneous approach (cont.)

Similarly, for the derivative of the boundary constraints with respect to x_K, we get

$$\frac{\partial r(x_0, x_K)}{\partial x_K} = \underbrace{\begin{bmatrix} \dfrac{\partial r_1(x_0, x_K)}{\partial x_K^{(1)}} & \dfrac{\partial r_1(x_0, x_K)}{\partial x_K^{(2)}} & \cdots & \dfrac{\partial r_1(x_0, x_K)}{\partial x_K^{(N_x)}} \\ \dfrac{\partial r_2(x_0, x_K)}{\partial x_K^{(1)}} & \dfrac{\partial r_2(x_0, x_K)}{\partial x_K^{(2)}} & \cdots & \dfrac{\partial r_2(x_0, x_K)}{\partial x_K^{(N_x)}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial r_{2N_r}(x_0, x_k)}{\partial x_K^{(1)}} & \dfrac{\partial r_{2N_r}(x_0, x_K)}{\partial x_K^{(2)}} & \cdots & \dfrac{\partial r_{2N_r}(x_0, x_K)}{\partial x_K^{(N_x)}} \end{bmatrix}}_{2N_r \times N_x}$$

For the inner product with the associated equality multiplier, we get

$$\underbrace{\underbrace{\frac{\partial r(x_0, x_K)}{\partial x_K}^T}_{N_x \times 2N_r} \underbrace{\lambda_{N_r}}_{2N_r \times 1}}_{N_x \times 1}$$

Problem formulations | Simultaneous approach (cont.)

$$\underbrace{E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right) + \left(\sum_{k=0}^{K-1} \lambda_{k+1}^T \left(f\left(x_k, u_k\right) - x_{k+1}\right) + \lambda_{N_r}^T r\left(x_0, x_K\right)\right)}_{\mathcal{L}(w,\lambda)}$$

The derivatives of the Lagrangian function with respect to the control variables u_k

- For any $k = 0, \dots, K-1$, we have

$$\nabla_{u_k} \mathcal{L}\left(w, \lambda\right) = \nabla_{u_k} L\left(x_k, u_k\right) + \frac{\partial f\left(x_k, u_k\right)}{\partial u_k}^T \lambda_{k+1}$$

Problem formulations | Simultaneous approach (cont.)

$$\begin{aligned} \nabla_w \mathcal{L}(w, \lambda) &= 0 \\ g(w) &= 0 \end{aligned}$$

We can collect all the KKT conditions and solve them using a Newton-type method

- The approach solves the problem in the full space of the decision variables

Problem formulations | Simultaneous approach (cont.)

The approach can be extended to more general discrete-time optimal control problems

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L_k(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f_k(x_k, u_k | \theta_x) = 0, && k = 0, 1, \dots, K-1 \\
& h_k(x_k, u_k) \leq 0, && k = 0, 1, \dots, K-1 \\
& R_K(x_K) + \sum_{k=0}^{K-1} r_k(x_k, u_k) = 0 \\
& h_K(x_K) \leq 0
\end{aligned}$$

All problem functions are explicitly time-varying and we have also a terminal inequality

- Moreover, the boundary conditions are expressed in general form

---

By collecting all variables in the vector w, we have the complete Lagrangian function

$$\mathcal{L}(w, \lambda, \mu) = f(w) + \lambda^T g(w) + \mu^T h(w)$$

The sequential approach

Problem formulations

Problem formulations | Sequential approach

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad && k = 0, 1, \dots, K-1 \\
& h(x_k, u_k) \leq 0, && k = 0, 1, \dots, K-1 \\
& r(x_0, x_N) = 0
\end{aligned}$$

The **sequential approach** solves the same task, in a reduced space of decision variables

The idea is to eliminate all the state variables $x_1, x_2, \dots, x_K$ by a forward-simulation

$$\begin{aligned}
x_0 &= x_0 \\
x_1 &= f(x_0, u_0) \\
x_2 &= f(x_1, u_1) \\
&= f(f(x_0, u_0), u_1) \\
x_3 &= f(x_2, u_2) \\
&= f(f(f(x_0, u_0), u_1), u_2) \\
\cdots &= \cdots \\
x_K &= \underbrace{f(f(f(x_0, u_0), u_1), u_2), \dots, u_{K-1})}_{\overline{x}_K(x_0, u_0, u_1, u_2, \dots, u_{K-1})}
\end{aligned}$$

Problem formulations | Sequential approach (cont.)

We can express the states as function of the initial condition and previous controls

$$
\begin{aligned}
x_0 &= \underbrace{x_0}_{\overline{x}_0(x_0)} \\
x_1 &= \underbrace{f\left(x_0, u_0\right)}_{\overline{x}_1(x_0,u_0)} \\
x_2 &= f\left(x_1, u_1\right) \\
&= \underbrace{f\left(f\left(x_0, u_0\right), u_1\right)}_{\overline{x}_2(x_0,u_0,u_1)} \\
x_3 &= f\left(x_2, u_2\right) \\
&= \underbrace{f\left(f\left(f\left(x_0, u_0\right), u_1\right), u_2\right)}_{\overline{x}_3(x_0,u_0,u_1,u_2)} \\
\cdots &= \cdots
\end{aligned}
$$

More generally, the dependence is on all the control variables and the initial condition

$$
\begin{aligned}
\overline{x}_0\left(x_o, u_0, u_1, \dots, u_{K-1}\right) &= x_0 \\
\overline{x}_{k+1}\left(x_0, u_0, u_1, \dots, u_{K-1}\right) &= f\left(\overline{x}_k\left(x_0, u_0, u_1, \dots, u_{K-1}\right), u_k\right), \quad k = 0, 1, \dots, K-1
\end{aligned}
$$

Problem formulations | Sequential approach

$$
\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right) \\
\text{subject to} \quad & x_{k+1} - f\left(x_k, u_k | \theta_x\right) = 0, \qquad k = 0, 1, \dots, K-1 \\
& h\left(x_k, u_k\right) \leq 0, \qquad k = 0, 1, \dots, K-1 \\
& r\left(x_0, x_N\right) = 0
\end{aligned}
$$

We can re-write the general discrete-time optimal control problem in such reduced form

$$
\begin{aligned}
\min_{\substack{x_0 \\ u_0, u_1, \dots, u_{K-1}}} \quad & E\left(\underbrace{\overline{x}_K\left(x_0, u_0, u_1, \dots, u_{K-1}\right)}\right) + \sum_{k=0}^{K-1} L\left(\underbrace{\overline{x}_k\left(x_0, u_0, u_1, \dots, u_{K-1}\right)}, u_k\right) \\
\text{subject to} \quad & h\left(\underbrace{\overline{x}_k\left(x_0, u_0, u_1, \dots, u_{K-1}\right)}, u_k\right) \leq 0, k = 0, 1, \dots, K-1 \\
& r\left(x_0, \underbrace{\overline{x}_K\left(x_0, u_0, u_1, \dots, u_{K-1}\right)}\right) = 0
\end{aligned}
$$

Problem formulations | Sequential approach (cont.)

$$\begin{aligned}\min_{\substack{x_0\\ u_0,u_1,\dots,u_{K-1}}} \quad & E\left(\overline{x}_K\left(x_0,u_0,u_1,\dots,u_{K-1}\right)\right)+\sum_{k=0}^{K-1} L\left(\overline{x}_k\left(x_0,u_0,u_1,\dots,u_{K-1}\right),u_k\right)\\ \text{subject to} \quad & h\left(\overline{x}_k\left(x_0,u_0,u_1,\dots,u_{K-1}\right),u_k\right)\leq 0, k=0,1,\dots,K-1\\ & r\left(x_0,\overline{x}_N\left(x_0,u_0,u_1,\dots,u_{K-1}\right)\right)=0\end{aligned}$$

The **objective function**, sum of **stage costs** $L\left(\overline{x}_k,u_k\right)$ and a **terminal cost** $E\left(\overline{x}_K\right)$

$$\underbrace{\sum_{k=0}^{K-1} L\left(\overline{x}_k,u_k\right)+E\left(\overline{x}_K\right)}_{f(w)\in\mathcal{R}}$$

That is,

$$L\left(x_0,u_0\right)+L\left(\overline{x}_1,u_1\right)+\cdots+L\left(\overline{x}_{K-1},u_{K-1}\right)+E\left(\overline{x}_K\right)$$

Stage cost is a (potentially nonlinear and time-varying) function of state and controls

The **decision variables**, $K\times N_u$ **control** and N_x **state variables**

$$\underbrace{\left(x_0\right)\cup\left(u_0,u_1,\dots,u_{K-1}\right)}_{w\in\mathcal{R}^{K\times N_u+N_x}}$$

Problem formulations | Sequential approach (cont.)

$$\begin{aligned}
\min_{\substack{x_0 \\ u_0, u_1, \ldots, u_{K-1}}} \quad & E\left(\overline{x}_K\left(x_0, u_0, u_1, \ldots, u_{K-1}\right)\right) + \sum_{k=0}^{K-1} L\left(\overline{x}_k\left(x_0, u_0, u_1, \ldots, u_{K-1}\right), u_k\right) \\
\text{subject to} \quad & h\left(\overline{x}_k\left(x_0, u_0, u_1, \ldots, u_{K-1}\right), u_k\right) \leq 0, k = 0, 1, \ldots, K-1 \\
& r\left(x_0, \overline{x}_N\left(x_0, u_0, u_1, \ldots, u_{K-1}\right)\right) = 0
\end{aligned}$$

The **equality constraints**, the N_r **boundary conditions**

$$\underbrace{r\left(x_0, \overline{x}_K\right) = 0}_{g(w) \in \mathcal{R}^{N_g}}$$

The **inequality constraints**

$$\underbrace{h\left(\overline{x}_k, u_k\right) \leq 0 \quad (k = 0, 1, \ldots, K-1)}_{h(w) \in \mathcal{R}^{N_h}}$$

Problem formulations | Sequential approach (cont.)

$$\begin{aligned}
\min_{\substack{x_0 \\ u_0, u_1, \dots, u_{K-1}}} \quad & E\left(\overline{x}_K\left(x_0, u_0, u_1, \dots, u_{K-1}\right)\right) + \sum_{k=0}^{K-1} L\left(\overline{x}_k\left(x_0, u_0, u_1, \dots, u_{K-1}\right), u_k\right) \\
\text{subject to} \quad & h\left(\overline{x}_k\left(x_0, u_0, u_1, \dots, u_{K-1}\right), u_k\right) \leq 0, k = 0, 1, \dots, K-1 \\
& r\left(x_0, \overline{x}_N\left(x_0, u_0, u_1, \dots, u_{K-1}\right)\right) = 0
\end{aligned}$$

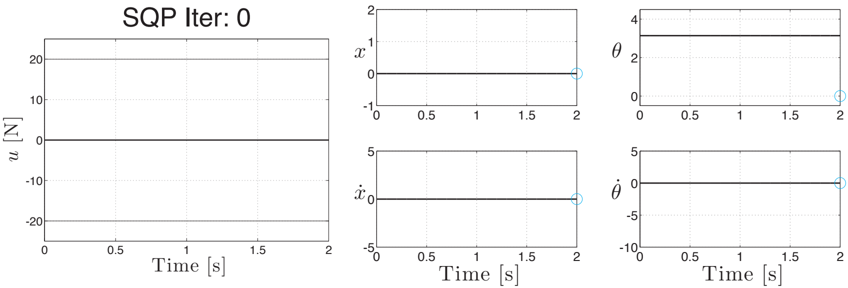
The Lagrangian function of the problem,

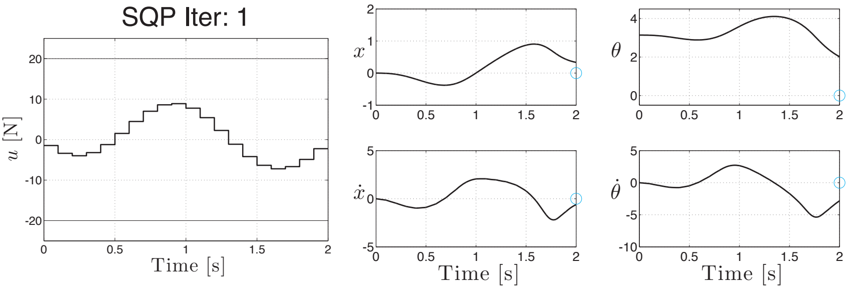
$$\mathcal{L}\left(w, \lambda, \mu\right) = f\left(w\right) + \lambda^T g\left(w\right) + \mu^T h\left(w\right)$$

The Karush-Kuhn-Tucker conditions,

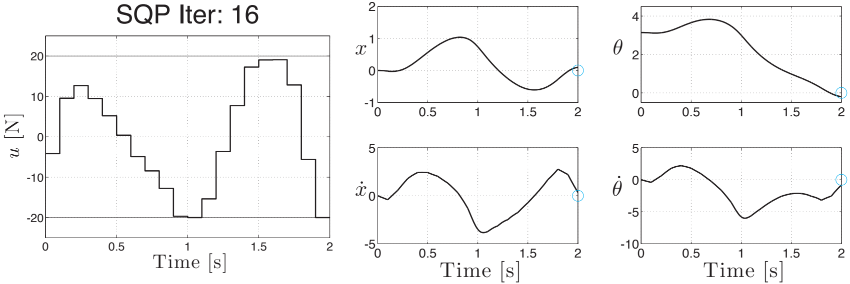
$$\begin{aligned}
\nabla f\left(w^*\right) - \nabla g\left(w^*\right)\lambda^* - \nabla h\left(w^*\right)\mu^* &= 0 \\
g\left(w^*\right) &= 0 \\
h\left(w^*\right) &\geq 0 \\
\mu^* &\geq 0 \\
\mu^*_{n_h} h_{n_h}\left(w^*\right) &= 0, \quad n_h = 1, \dots, N_h
\end{aligned}$$

Problem formulations | Sequential approach (cont.)





Problem formulations | Sequential approach (cont.)



For computational efficiency, it is preferable to use specific structure-exploiting solvers

- Such solvers recognise the sparsity properties of this class of problems