

EEN-E2001 Computational Fluid Dynamics

Lecture 3: Navier-Stokes equation and pressure

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January 29th 2024

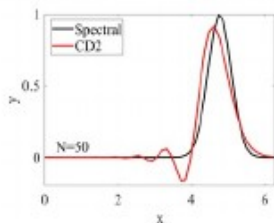
Aalto University, School of Engineering

Lecture 1: Linear PDEs and finite difference method

HW1

$$\frac{\partial T}{\partial t} + \nabla \cdot T \mathbf{u} = \alpha \nabla^2 T$$

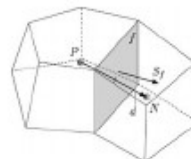
$$\frac{\partial T}{\partial x} \approx \frac{T_{i+1} - T_{i-1}}{2\Delta x}$$



Lecture 2: Gauss' theorem and finite volume method

$$\int_{\Omega} \nabla \cdot (T \mathbf{u}) d\Omega = \int_{\partial\Omega} (T \mathbf{u}) \cdot \mathbf{n} dS$$

$$\int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} dS \approx \sum_f \mathbf{u}_f \cdot \mathbf{n}_f dS_f$$

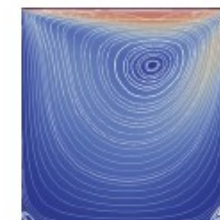


Lecture 3: Navier-Stokes equation and pressure

HW2

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u} \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}$$

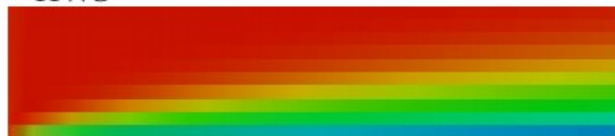
$$-\nabla^2 p = \nabla \cdot \nabla \cdot \mathbf{u} \mathbf{u}$$



Lecture 4: OpenFOAM code and structure

HW3

```
fvVectorMatrix UEqn
(
    fvm::ddt(U)
    + fvm::div(phi, U)
    - fvm::laplacian(nu, U)
);
```



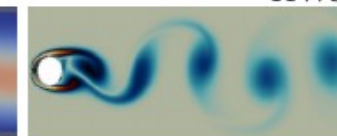
Lecture 5: Fluid physical phenomena: (laminar and turbulent flow)

HW4



Lecture 6: Matrix equations $Ax=b$ and final assignment

HW5



Intended learning objectives of the full lecture

After the lecture the student:

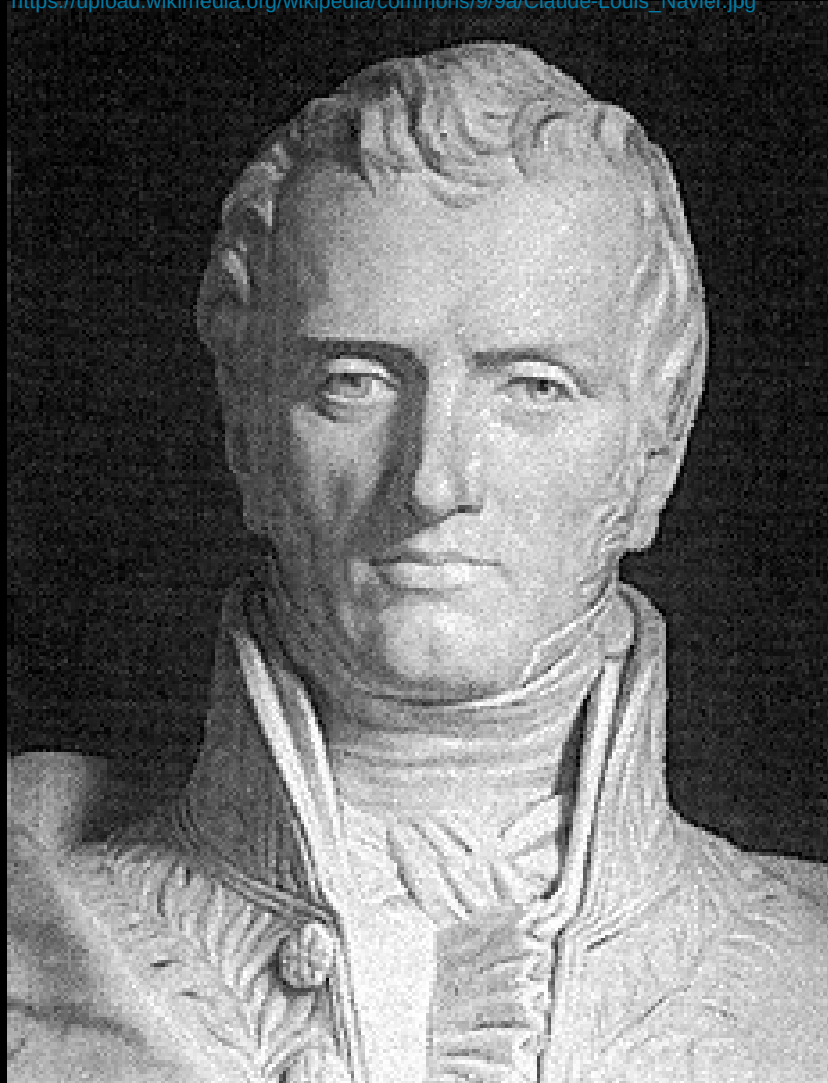
- Can explain what is the role of pressure in incompressible flows.
- Can explain the basic idea how Navier-Stokes equation is discretized in fvm using projection method and explicit time integration.

Navier-Stokes equations

Clay Institute: Among the 7 most important unsolved problems in math

https://en.wikipedia.org/wiki/Navier–Stokes_equations

https://upload.wikimedia.org/wikipedia/commons/9/9a/Claude-Louis_Navier.jpg



<https://upload.wikimedia.org/wikipedia/commons/thumb/a/ad/Ggstokes.jpg/300px-Ggstokes.jpg>



CFD simulation and PDE solution includes at least the following aspects covered on the course

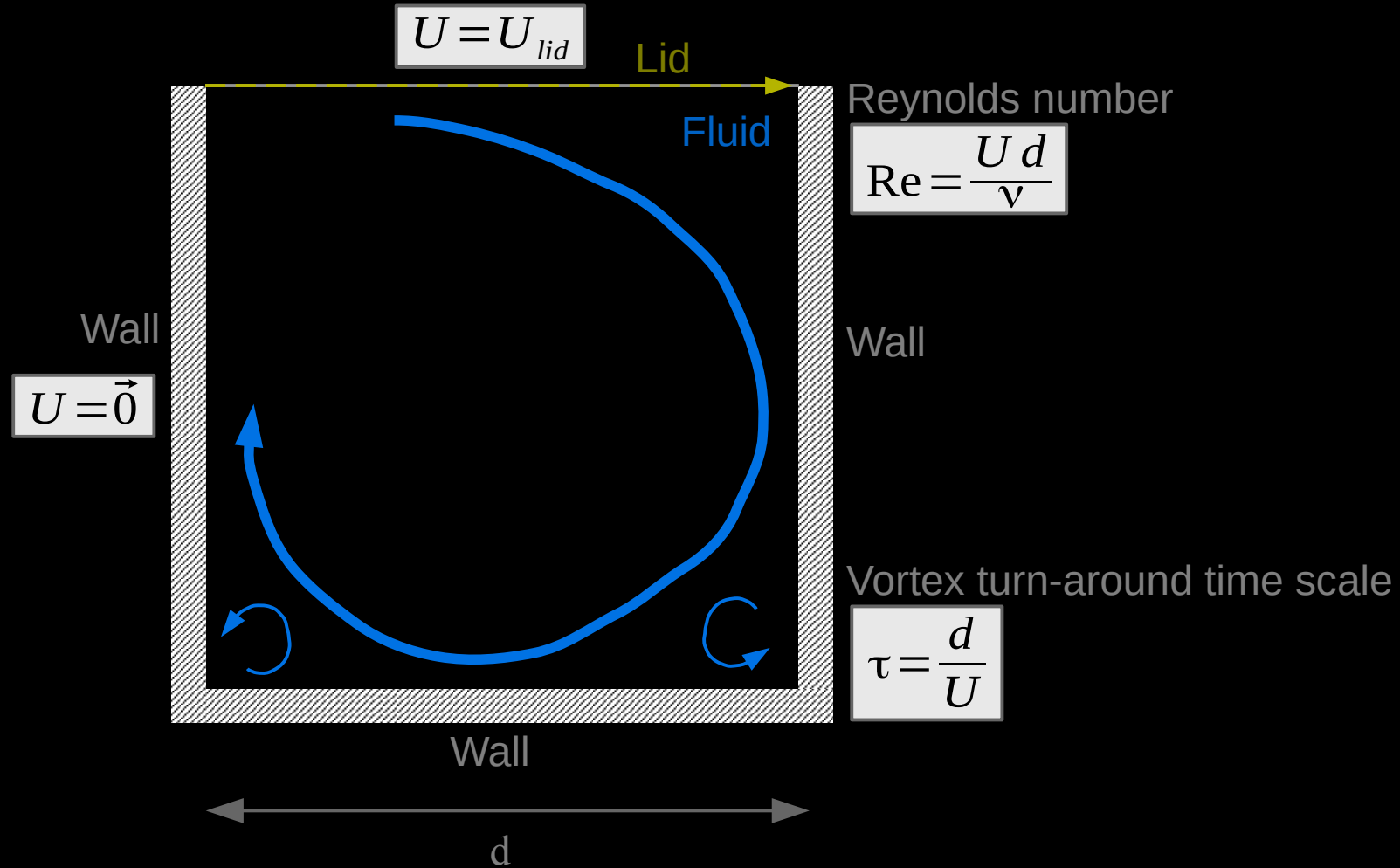
- 1) **Physics** identification.
- 2) **Mathematical equations and physics interpretation.**
Boundary/initial conditions.
- 3) **Objectives, feasibility, and time-constraints.**
- 4) **Numerical method and modeling assumptions.**
- 5) **Geometry and mesh generation.**
- 6) **Computing** i.e. running simulation.
- 7) **Visualization and post-processing.**
- 8) **Validation and verification, reference data.** Reporting, analysis and discussion of the results. Are the results sane?

Motivational aspects to study flow in a cavity

Relevance: HW2

Homework 2: Lid driven cavity

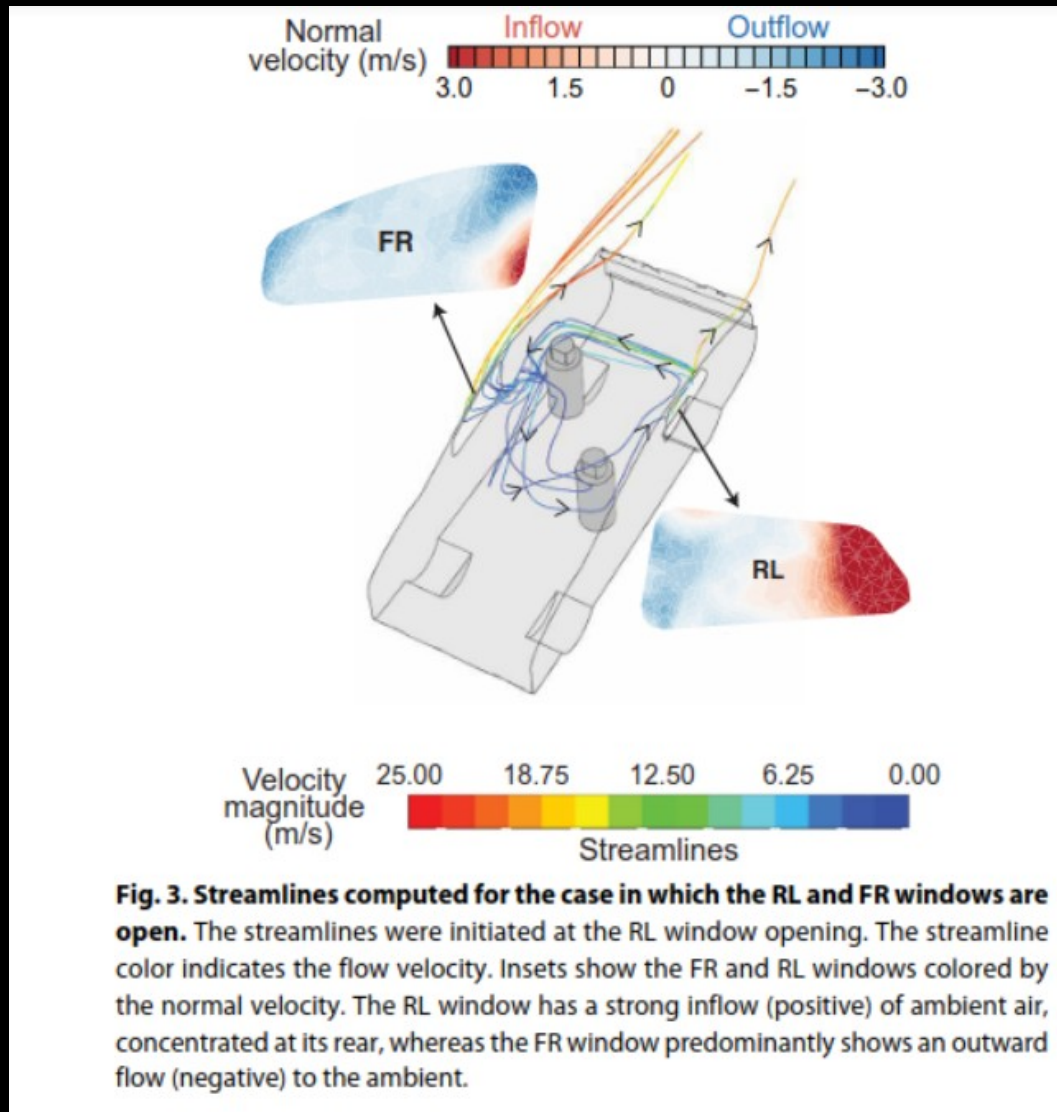
Moving lid creates a vortex in a box



Motivation: window induced airflow in a car is a simple example of “flow in a cavity”

Mathai et al. (2021)

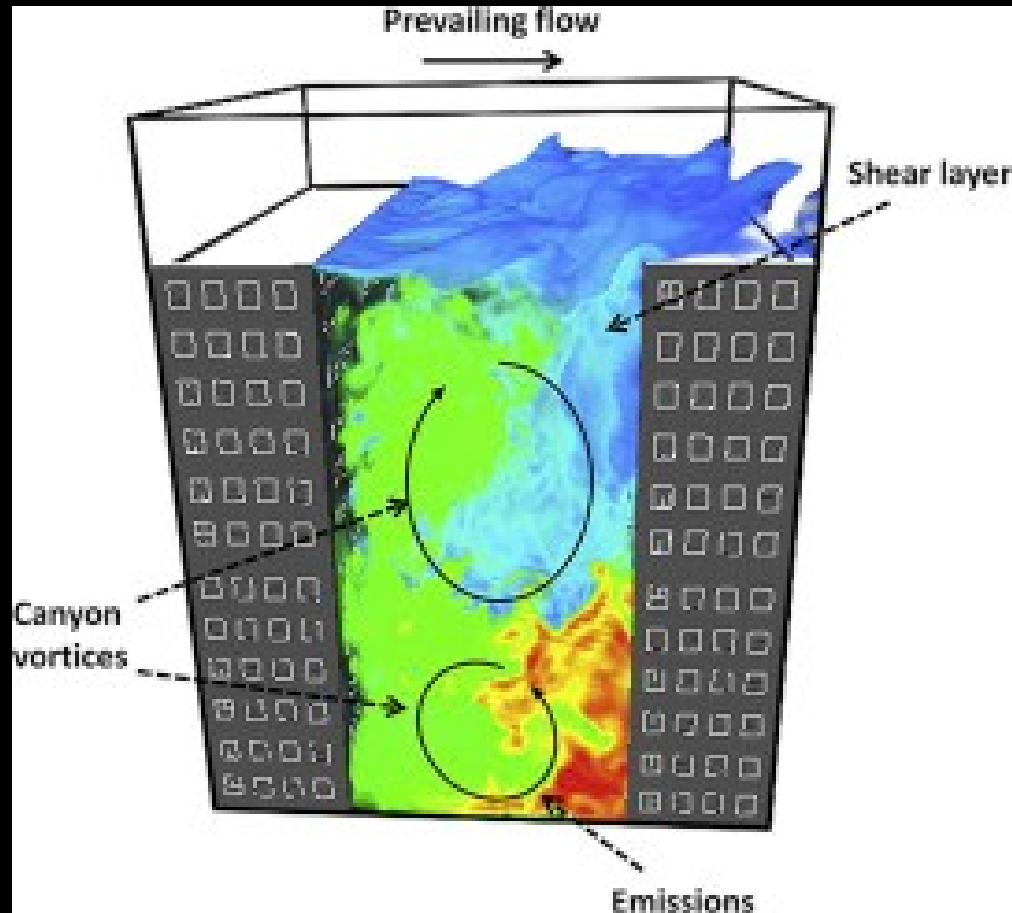
<https://www.science.org/doi/epdf/10.1126/sciadv.abe0166>



Motivation: another example for “flow in a cavity” could be wind flow pattern between two buildings

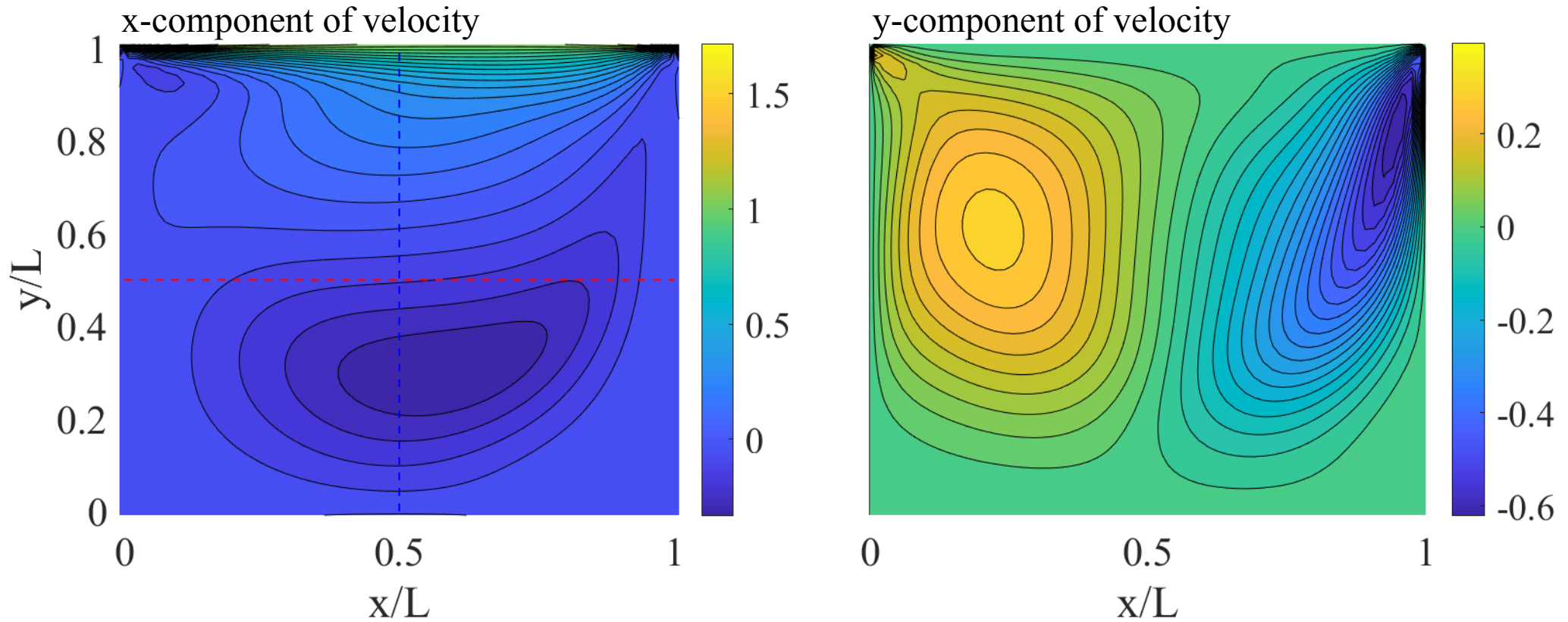
Zhong et al. (2017)

<https://www.sciencedirect.com/science/article/abs/pii/S0269749116312398#undfig1>



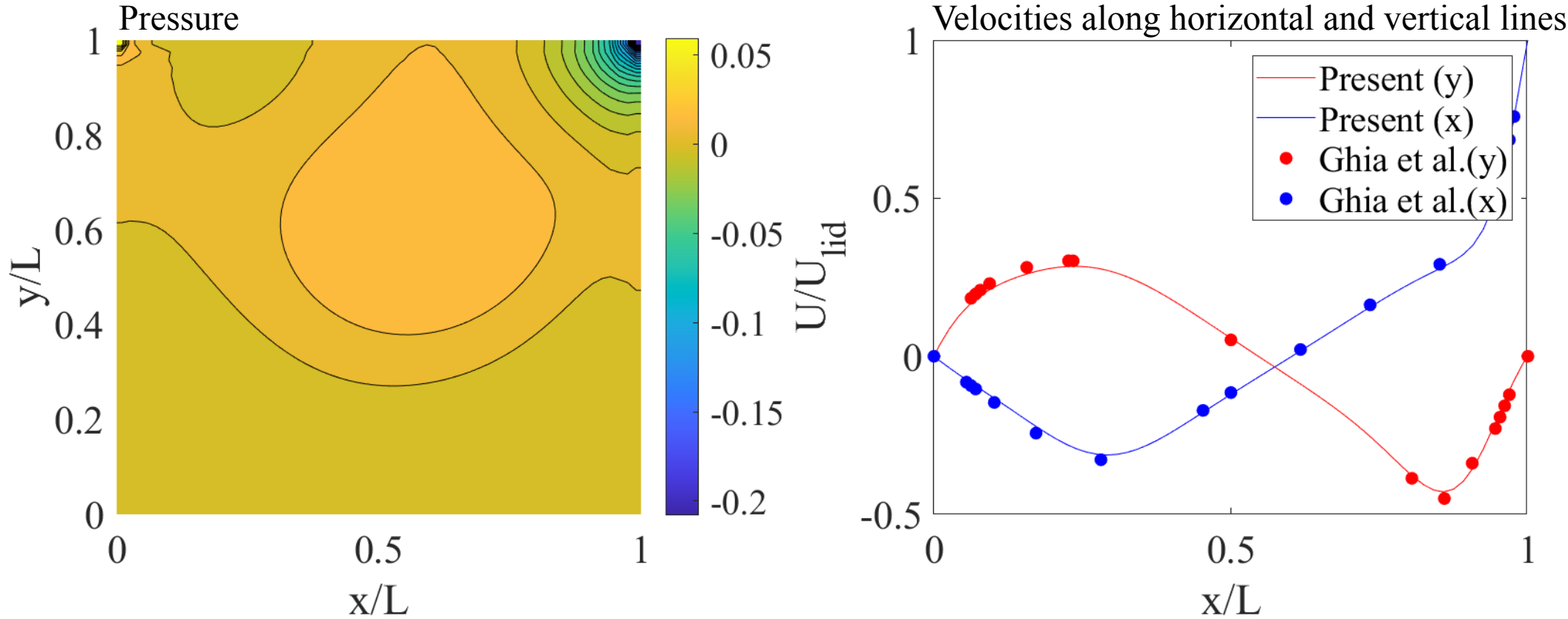
Homework 2: Lid driven cavity

Comparison of a Navier-Stokes simulation against reference data by Ghia et al. Here: $Re=100$ & resolution 60^2 points.



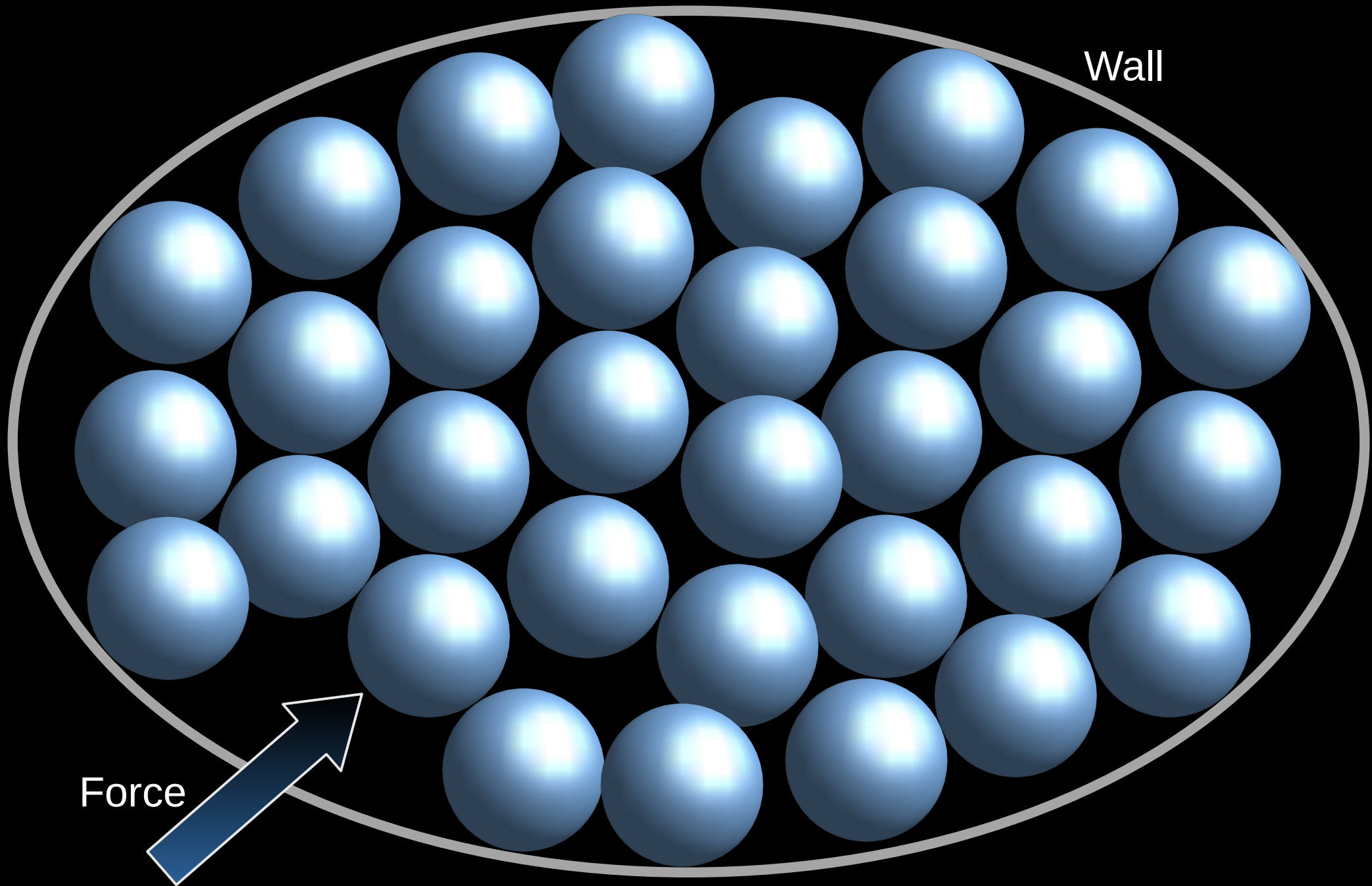
Homework 2: Lid driven cavity

Comparison of a Navier-Stokes simulation against reference data by Ghia et al.

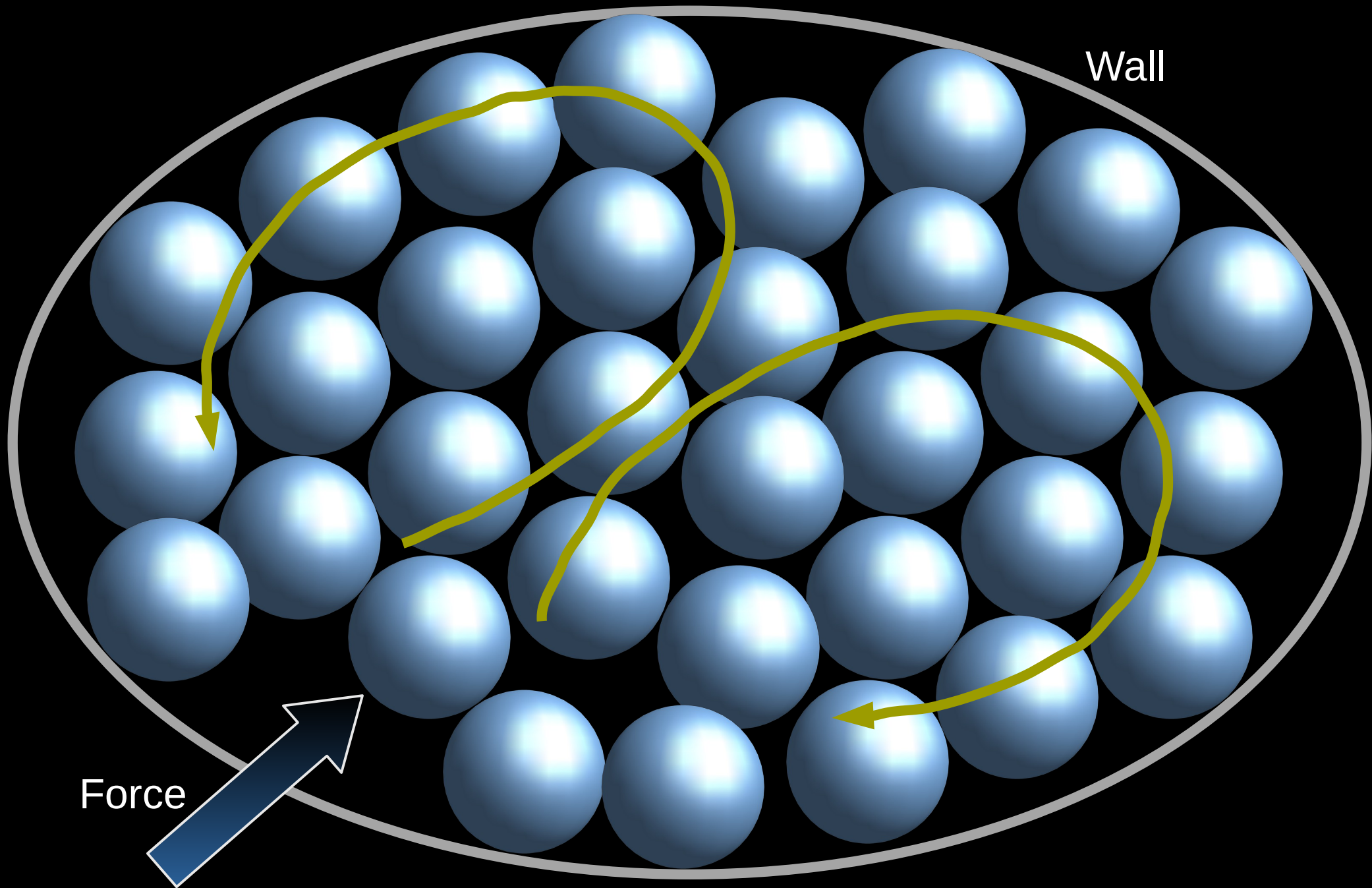


Incompressible Navier-Stokes equation

Rigid sphere analogy and incompressible flows: what happens when one of the spheres is pushed?



A vortical motion appears when all other spheres “feel” the force while the wall keeps the spheres confined in the container. This is called “**elliptical**” or “**global**” character. In incompressible flows pressure is solved from **Poisson eqn** which is of such elliptical character: when fluid element moves all other elements will **immediately** feel that motion via the pressure changes.



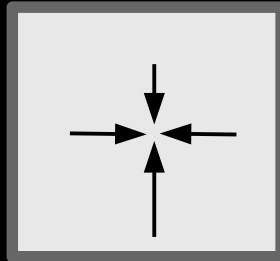
Compressible and incompressible flows

Continuity equation tells that mass is conserved (ρ = density):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{u} \rho) = 0$$

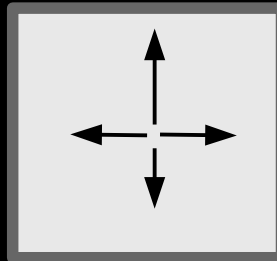
Velocity vectors in a given point

$$\frac{\partial \rho}{\partial t} > 0, \nabla \cdot (\vec{u} \rho) < 0$$



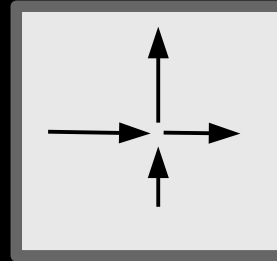
Compression

$$\frac{\partial \rho}{\partial t} < 0, \nabla \cdot (\vec{u} \rho) > 0$$



Expansion

$$\frac{\partial \rho}{\partial t} = 0, \nabla \cdot \vec{u} = 0$$



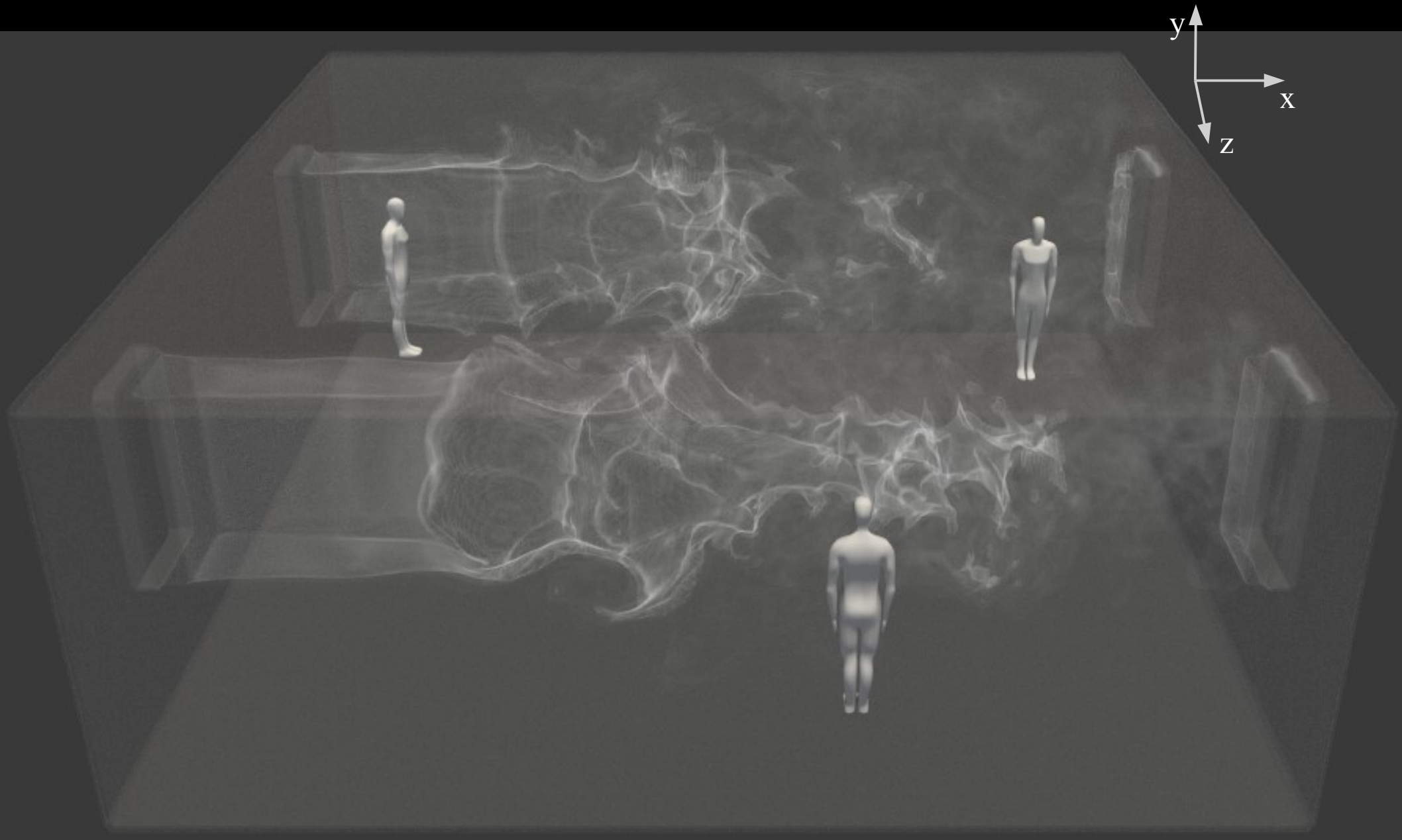
Incompressible

The incompressibility condition: at each point massflow entering a control volume = mass flow exiting the control volume (ρ = constant).

$$\nabla \cdot \vec{u} = 0$$

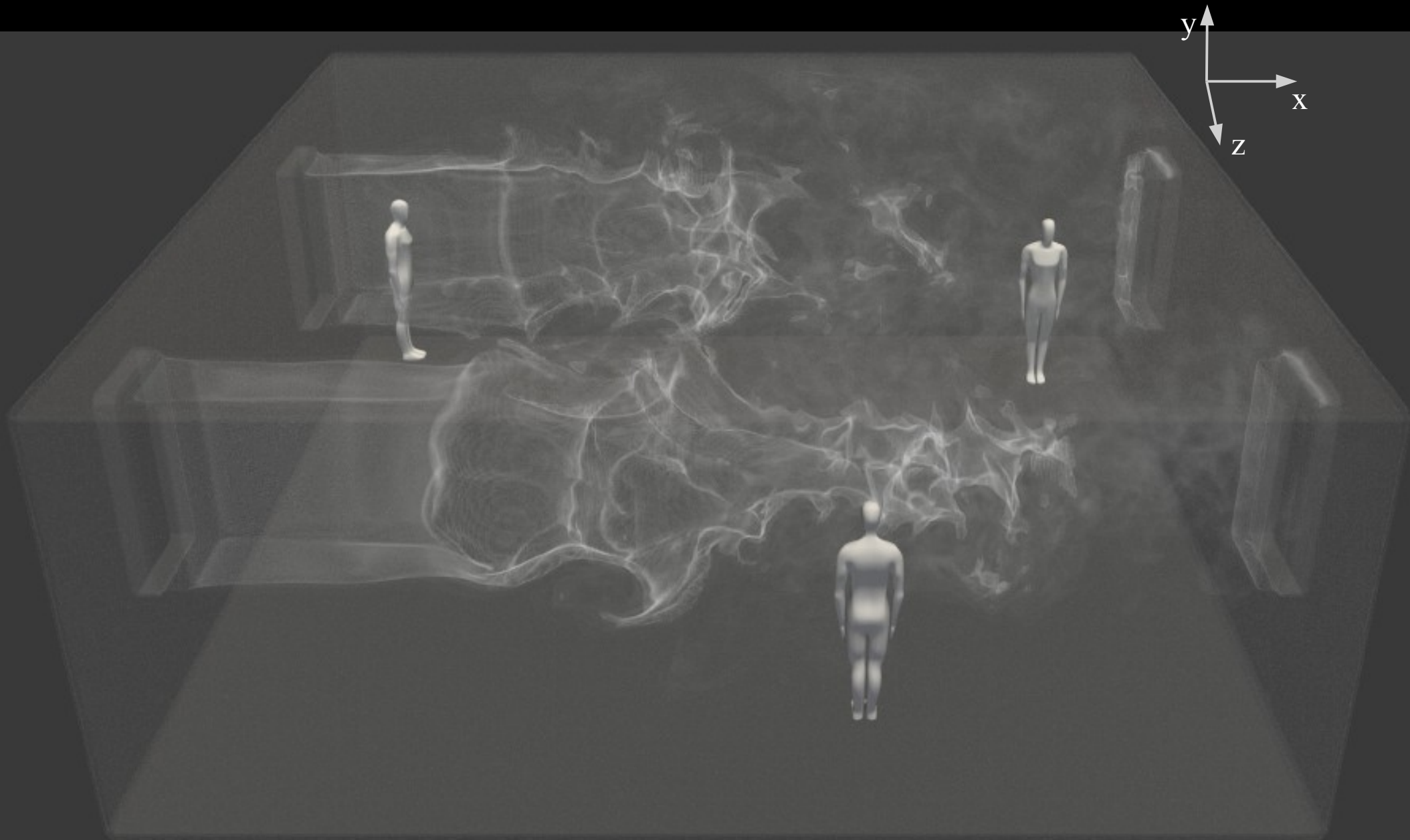
Gauss theorem applied on the mass conservation law:

$$\int_{V_{room}} \nabla \cdot \vec{u} dV = \int_{A_{room}} \vec{u} \cdot \vec{n} dA = 0$$



The standard mass conservation law is commonly written on B.Sc./M.Sc. courses as $U_1 A_1 + U_2 A_2 = U_3 A_3 + U_4 A_4$. Since $-n_{\text{left}} = n_{\text{right}}$ we get:

$$\int_{A_{\text{win}1}} \vec{u} \cdot \vec{n} dA + \int_{A_{\text{win}2}} \vec{u} \cdot \vec{n} dA = \int_{A_{\text{win}3}} \vec{u} \cdot \vec{n} dA + \int_{A_{\text{win}4}} \vec{u} \cdot \vec{n} dA$$



Incompressible Navier-Stokes equation

Convection form

i^{th} component of velocity

Kinematic viscosity [m²/s]

$$\frac{\partial u_i}{\partial t} + \vec{u} \cdot \nabla u_i = -\nabla p + \nu \nabla^2 u_i$$

Note:
essentially
same eqn

Convection-diffusion eqn

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \nabla c = \alpha \nabla^2 c$$

Pressure

$$\nabla \cdot \vec{u} = 0$$

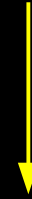
Incompressible Navier-Stokes equation

Conservative form

i^{th} component of velocity



Kinematic viscosity [m²/s]



$$\frac{\partial u_i}{\partial t} + \nabla \cdot \vec{u} u_i = -\nabla p + \nu \nabla^2 u_i$$

Convection-diffusion eqn

$$\frac{\partial c}{\partial t} + \nabla \cdot \vec{u} c = \alpha \nabla^2 c$$



Pressure

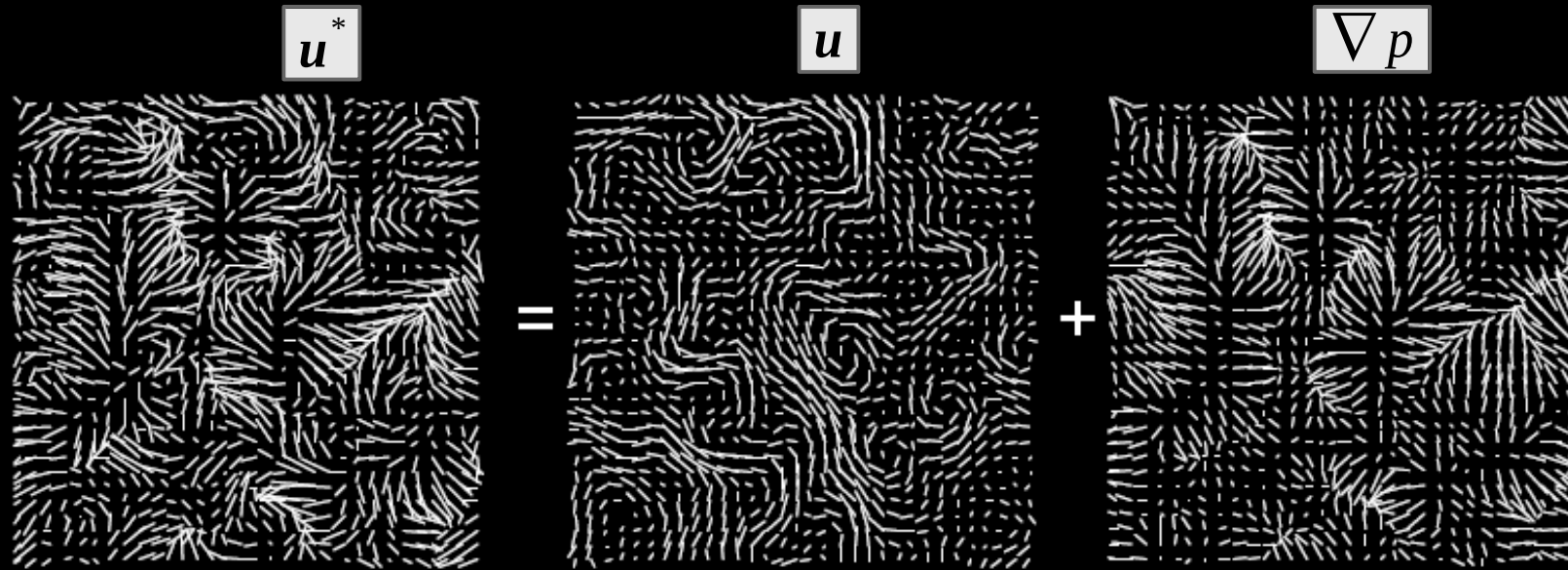
$$\nabla \cdot \vec{u} = 0$$

Pressure in incompressible flows

Relevance: Non-linear convection term creates divergence to velocity which must be “removed” by pressure gradient to obtain mass balance and fulfill $\text{div}(\mathbf{u}) = 0$ condition

Vector fields can be divided into two parts via “Helmholtz-Hodge” decomposition

Conservation of Mass



Our field = mass conserving + gradient

Hodge decomposition

Mathematical background for the projection method

Vectors: C=convection term, D=diffusion term, P=pressure gradient

The core problem in solving incompressible Navier-Stokes method:

We would like to find a $\Delta u_i = \Delta t(-C+D+P)$ to update solution at each grid point as $u_i^{n+1} = u_i^n + \Delta u_i$. We can estimate $C=C(x,y,z,t)$ & $D=D(x,y,z,t)$ ($t=n\Delta t$). But how do we know the pressure gradient $P=P(x,y,z,t)$?

Helmholtz-Hodge theorem: any vector field \mathbf{u}^* can be expressed as a sum of fields as follows (B.Sc. math):

$$\mathbf{u}^* = \mathbf{u} + \nabla p, \text{ where } \nabla \cdot \mathbf{u} = 0 \text{ and } \nabla \times \nabla p = 0$$

Consequence of Helmholtz-Hodge theorem: the function p is a solution of the Poisson eqn. If we could find \mathbf{u}^* then we can also solve p .

$$\nabla^2 p = \nabla \cdot \mathbf{u}^*$$

Derivation of Poisson equation for pressure 1/2

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

Take divergence from both sides of NS-eqn

$$\frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_i} \frac{\partial u_j u_i}{\partial x_j} = -\frac{\partial^2 p}{\partial x_i^2} + \nu \frac{\partial^2}{\partial x_j^2} \frac{\partial u_i}{\partial x_i}$$

Derivation of Poisson equation for pressure 2/2

Incompressible flow
→ $\text{div}(\mathbf{u})=0$

Incompressible flow
→ $\text{div}(\mathbf{u})=0$

$$\frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_i} \frac{\partial u_j u_i}{\partial x_j} = - \frac{\partial^2 p}{\partial x_i^2} + \nu \frac{\partial^2}{\partial x_j^2} \frac{\partial u_i}{\partial x_i}$$

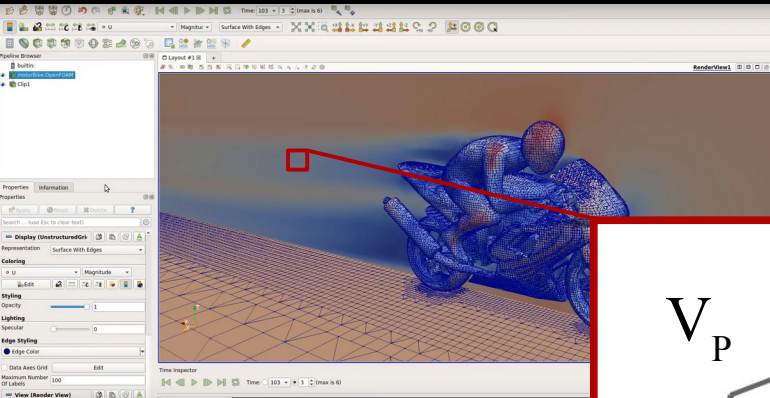
Poisson equation for pressure

$$- \frac{\partial^2 p}{\partial x_i^2} = \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

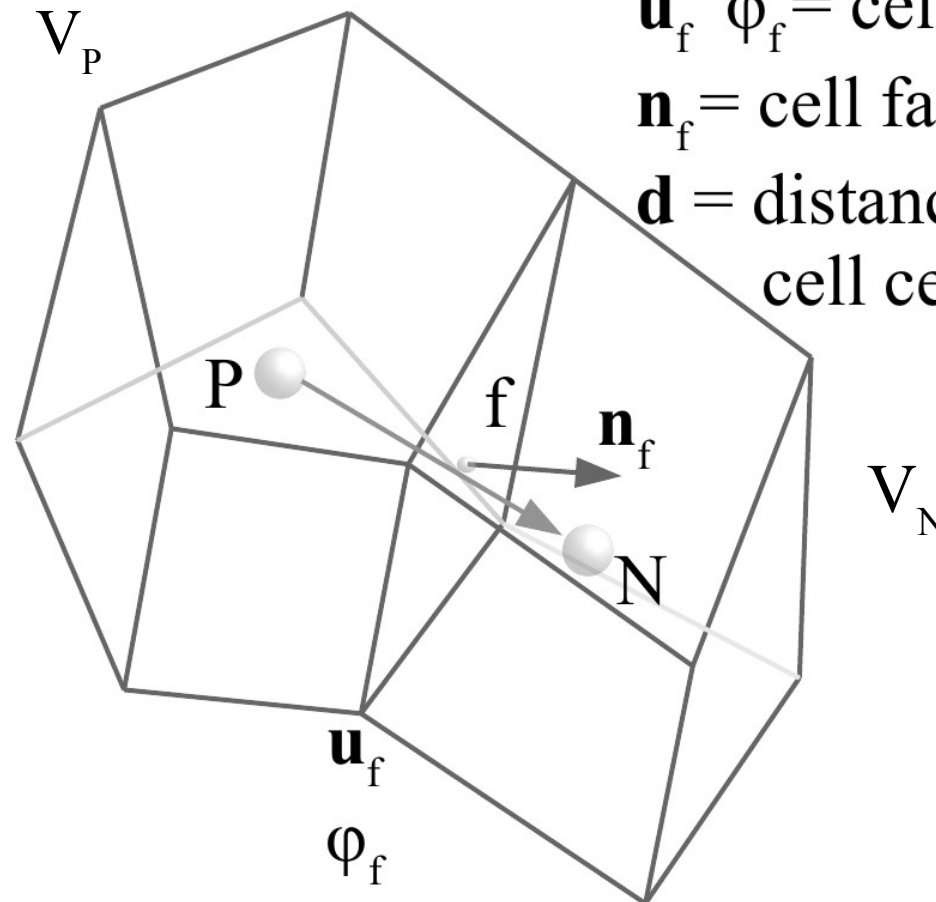
$$- \nabla^2 p = \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

In practice Poisson equation is a matrix equation which is solved with a linear system solver for $\mathbf{A} \mathbf{x} = \mathbf{b}$ (more later in the course).

Basic idea of finite volume method: divide geometry into small volumes and update numerical solution at **volume centroids** by estimating e.g. **mass & momentum fluxes** through the **faces** of the **control volumes** during small time intervals



Control volume



$\mathbf{u}_f \phi_f =$ cell face interpolants
 $\mathbf{n}_f =$ cell face normal vector
 $\mathbf{d} =$ distance vector between cell centroids P and N

Projection method in a nutshell

Projection method

Step 1: Calculate velocity prediction \mathbf{u}^* using only \mathbf{C} and \mathbf{D} from previous timestep

$$\mathbf{u}^* = \mathbf{u}_n + \Delta t (\mathbf{D} - \mathbf{C})$$

Step 2: Calculate divergence of \mathbf{u}^* and use it as source term in Poisson equation to find p using a standard Poisson solver.

$$\nabla^2 p = \nabla \cdot \mathbf{u}^*$$

Step 3: Calculate pressure gradient via e.g. CD2 and do pressure correction (projection)

$$\mathbf{u}_{n+1} = \mathbf{u}^* - \nabla p$$

Step 4: Based on Helmholtz-Hodge theorem \mathbf{u}_{n+1} is incompressible and mass conserving:

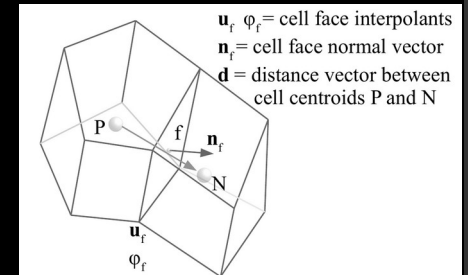
$$\nabla \cdot \mathbf{u}_{n+1} = 0$$

Finite volume method in a nutshell for Navier-Stokes eqn and projection method. Discretization for Steps 1-3 is discussed below:

Step 1: fvm estimation of convection $C_i=C_i(x,y,z,t)$ and diffusion $D_i=D_i(x,y,z,t)$ terms in Navier-Stokes eqn to find the prediction u^* ($t=n\Delta t$):

$$C_i = \frac{1}{V} \int_V \nabla \cdot (\mathbf{u} u_i) dV = \frac{1}{V} \int_A (\mathbf{u} u_i) \cdot \mathbf{n} dA \approx \frac{1}{V} \sum_{faces} (\mathbf{u}_f u_f^i) \cdot \mathbf{n}_f dA_f$$

$$D_i = \frac{1}{V} \int_V \nu \nabla \cdot \nabla u_i dV = \frac{1}{V} \int_A \nu \nabla u_i \cdot \mathbf{n} dA \approx \frac{1}{V} \sum_{faces} \nu \nabla u_f^i \cdot \mathbf{n}_f dA_f$$



Step 2a: fvm estimation of divergence of u^*

$$\nabla \cdot \mathbf{u}^* \approx \frac{1}{V} \int_V \nabla \cdot \mathbf{u}^* dV \approx \frac{1}{V} \sum_{faces} \mathbf{u}_f \cdot \mathbf{n}_f dA_f$$

Step 2b: fvm estimation of matrix eqn for pressure (more on matrices later on)

$$\Delta p \approx \frac{1}{V} \int_V \nabla \cdot \nabla p dV = \frac{1}{V} \int_A \nabla p \cdot \mathbf{n} dA \approx \frac{1}{V} \sum_{faces} \nabla p_f \cdot \mathbf{n}_f dA_f$$

Step 3: fvm estimation of pressure gradient

$$\nabla p \approx \frac{1}{V} \int_V \nabla p dV = \frac{1}{V} \int_A p \mathbf{n}_i dA_i \approx \frac{1}{V} \sum_{faces} p_f^i \mathbf{n}_f^i dA_f^i$$