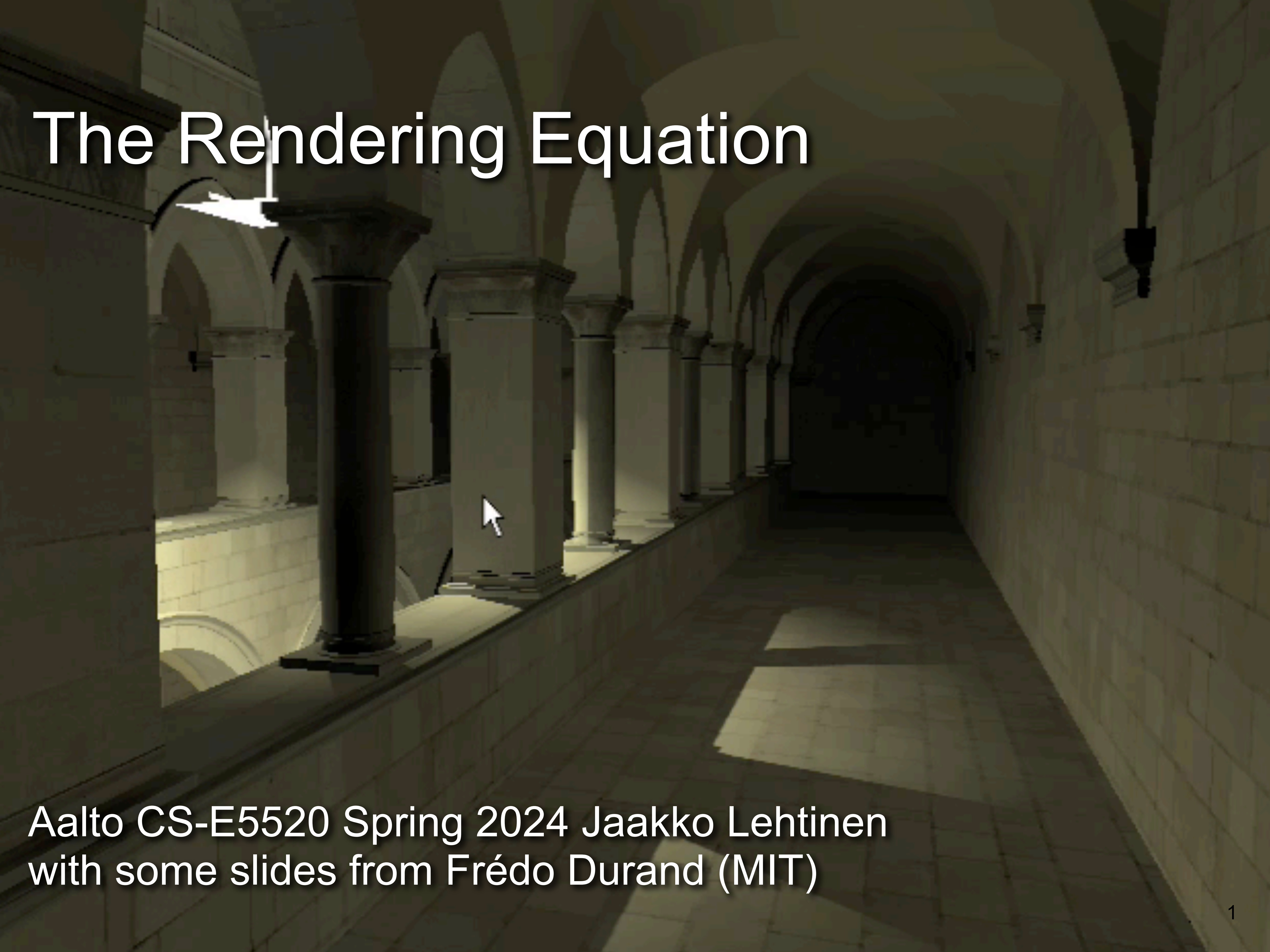
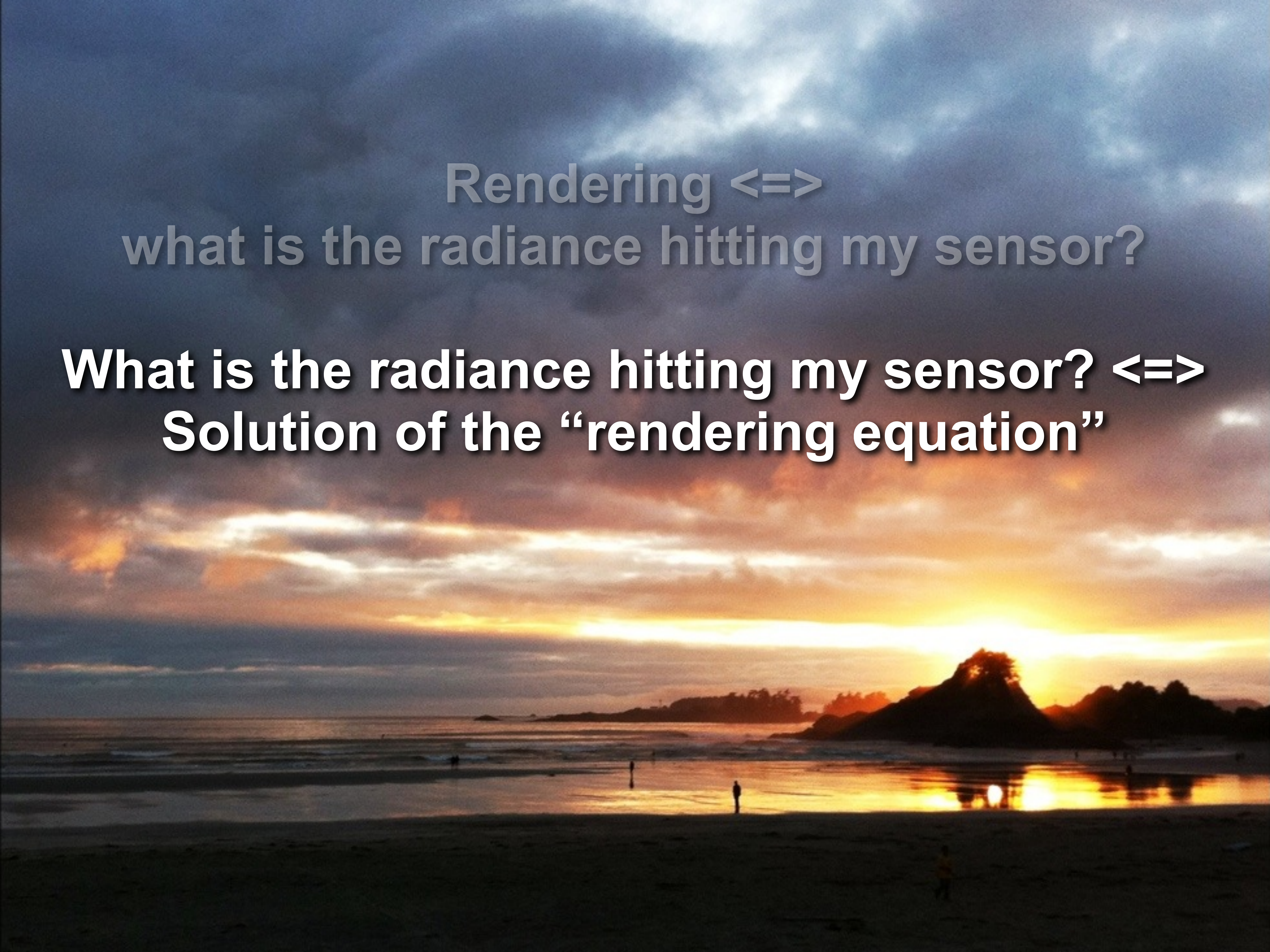
The Rendering Equation



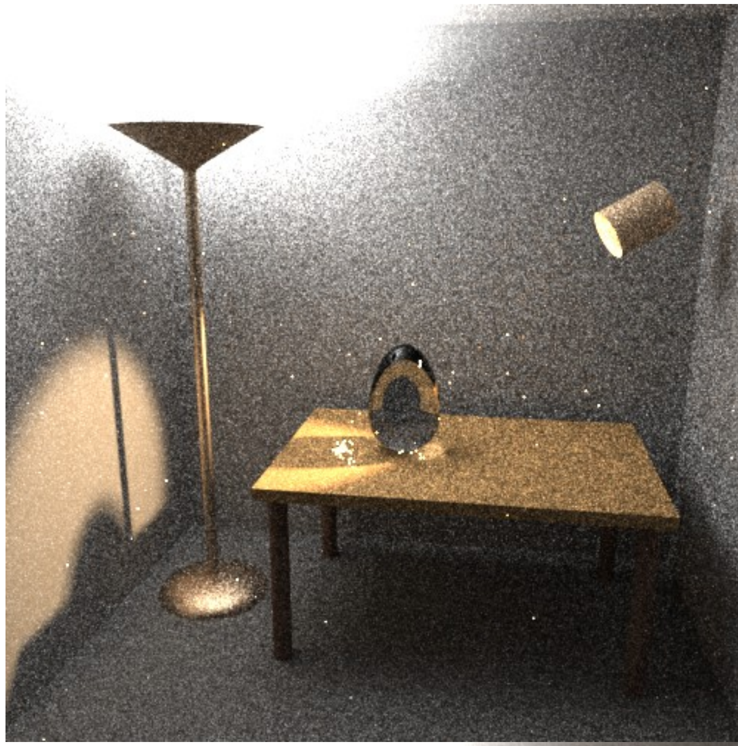
Aalto CS-E5520 Spring 2024 Jaakko Lehtinen
with some slides from Frédo Durand (MIT)

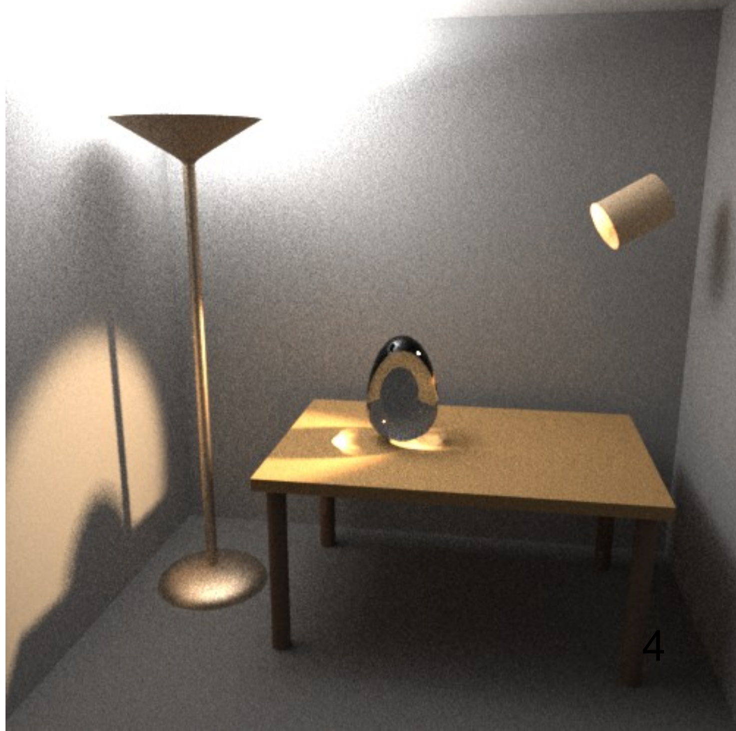

Rendering <=>
what is the radiance hitting my sensor?


Rendering <=>
what is the radiance hitting my sensor?
What is the radiance hitting my sensor? <=>
Solution of the “rendering equation”

Today

- Global Illumination
  - Rendering Equation
  - Gets us indirect lighting
- Next time
  - Monte Carlo integration
  - Better sampling
    - importance
    - stratification





Recap: Reflectance Equation

$$L(x \to \mathbf{v}) =$$

outgoing radiance

$$\int_\Omega f_r(x, \mathbf{l} \to \mathbf{v}) \boxed{L(x \leftarrow \mathbf{l}) \; \cos\theta} \, \mathrm{d}\mathbf{l}$$

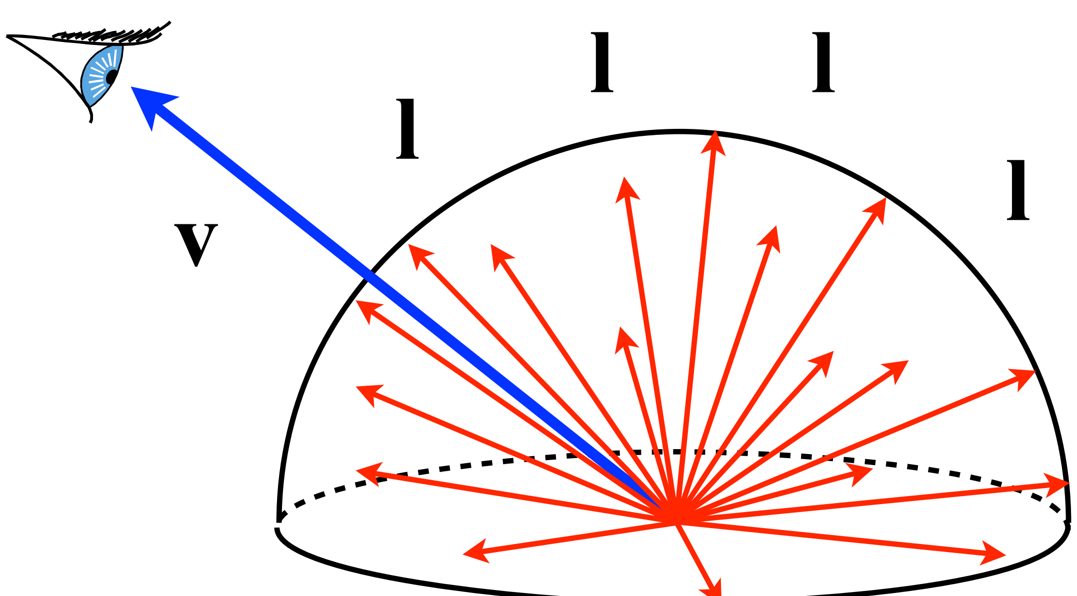
L_in * cos = incident differential irradiance

integral over hemisphere

BRDF

incoming radiance

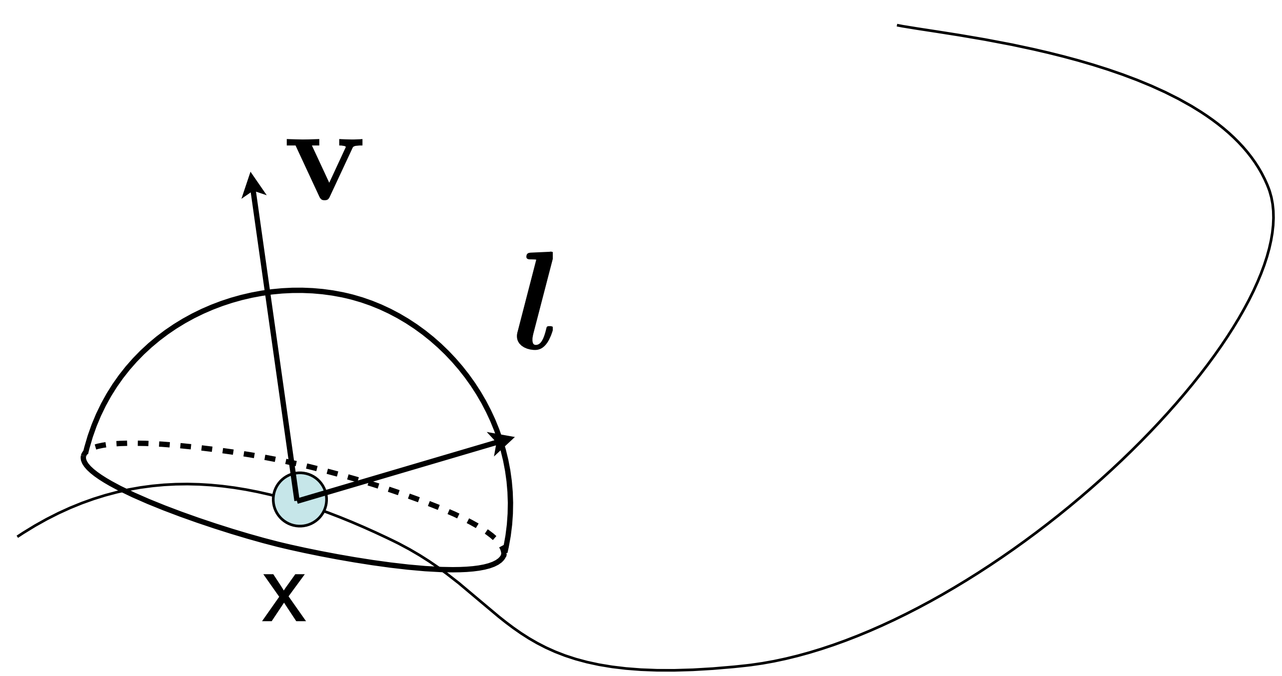
cosine of incident angle



The Way To Global Illumination

$$L(x \rightarrow \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

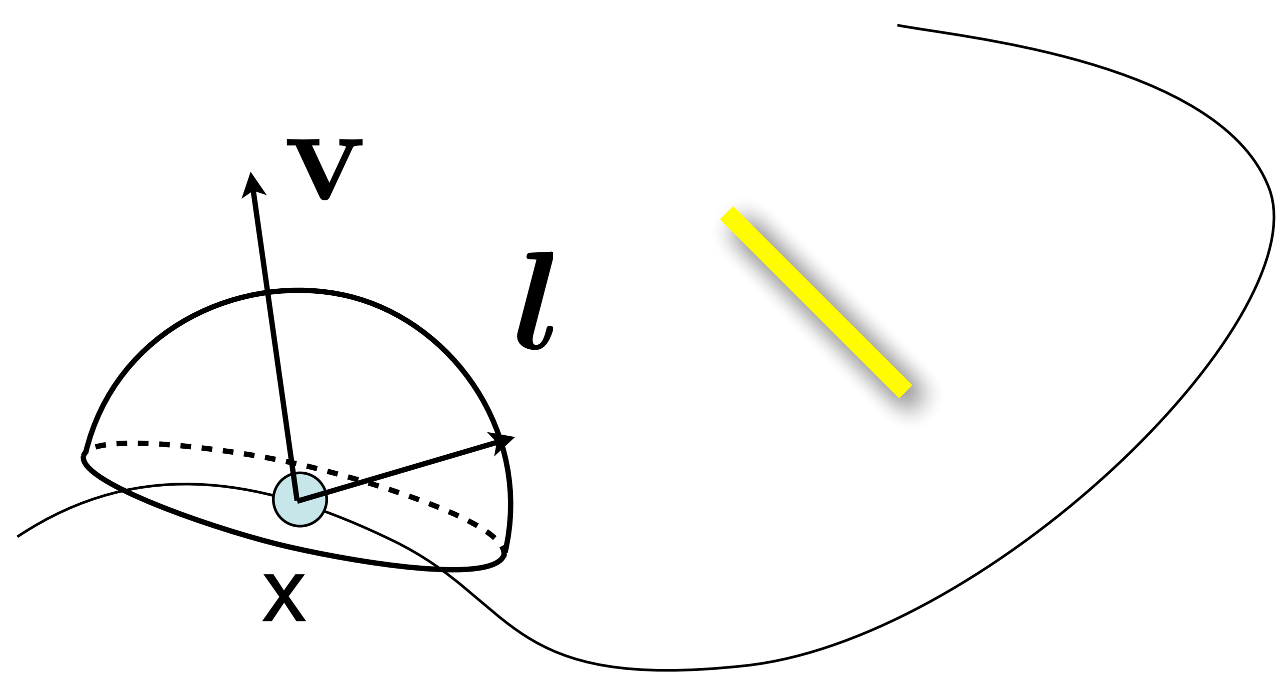
- Where does incident radiance L come from?



The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

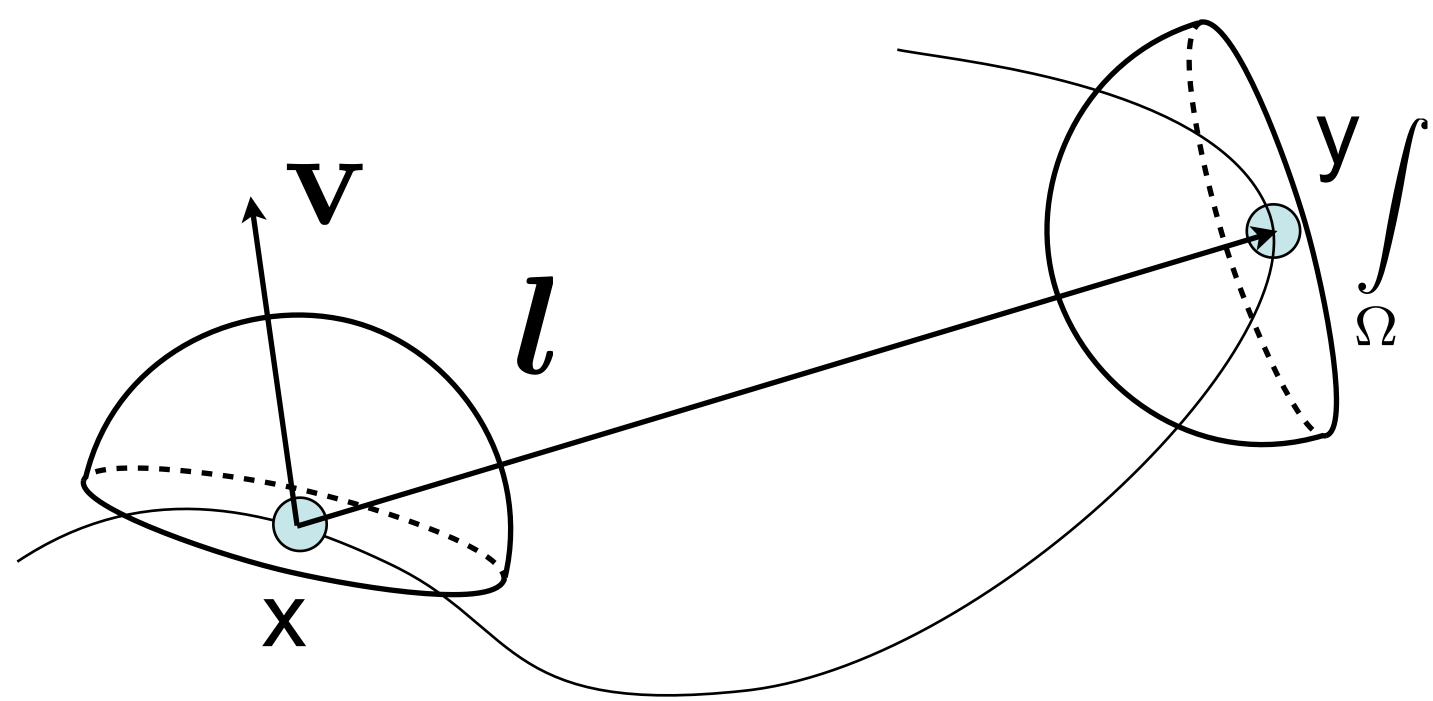
- Where does incident radiance L come from?
- Familiar case: From a light source, for certain **l**
  - But what about other incident directions?



The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \,\mathrm{d}\mathbf{l}$$

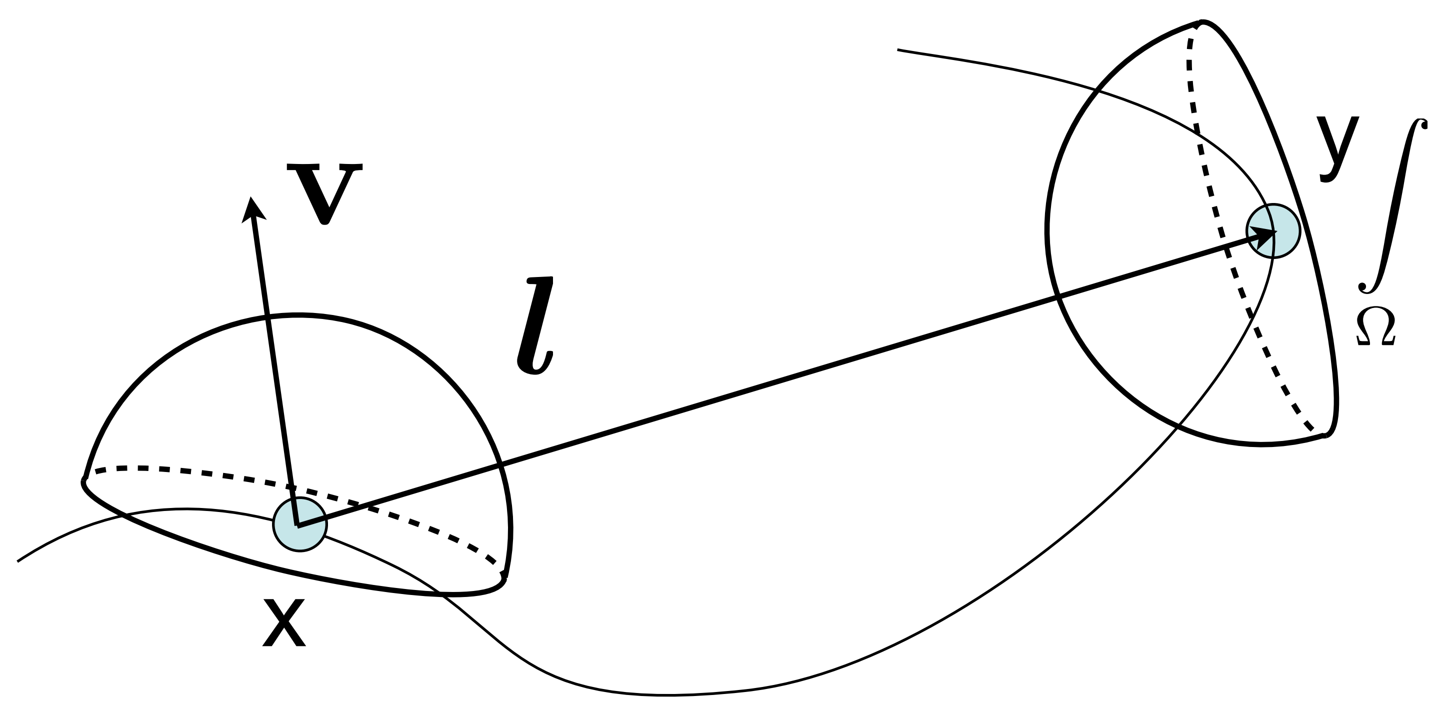
- Where does incident radiance L come from?
  - It is the light reflected towards *x* from the surface point *y* in direction ***l***



The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

- Where does incident radiance L come from?
  - It is the light reflected towards *x* from the surface point *y* in direction ***l*** ==> must compute similar integral for every ***l***!
    - Recursive!

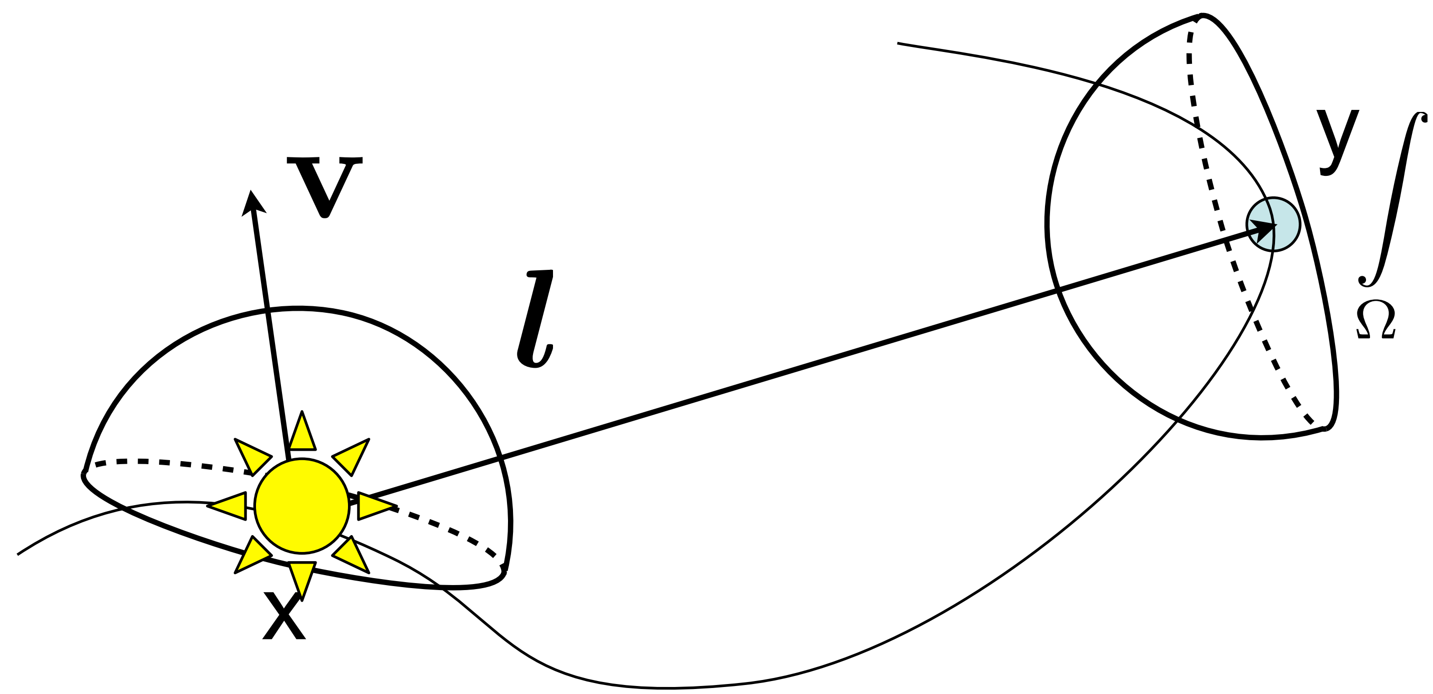


Rendering Equation

$$L(x \rightarrow \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta\, \mathrm{d}\mathbf{l}$$

$$\boxed{+\ E(x \rightarrow \mathbf{v})}$$

- Where does incident radiance L come from?
  - It is the light reflected towards *x* from the surface point *y* in direction ***l*** ==> must compute similar integral for every ***l***!
    - Recursive!
- ...and if *x* happens to be on a light source, we add its emitted contribution *E*



The Rendering Equation

$$L(x \rightarrow \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta\, \mathrm{d}\mathbf{l}$$
$$+ E(x \rightarrow \mathbf{v})$$

- Let's bask in its glory for a moment

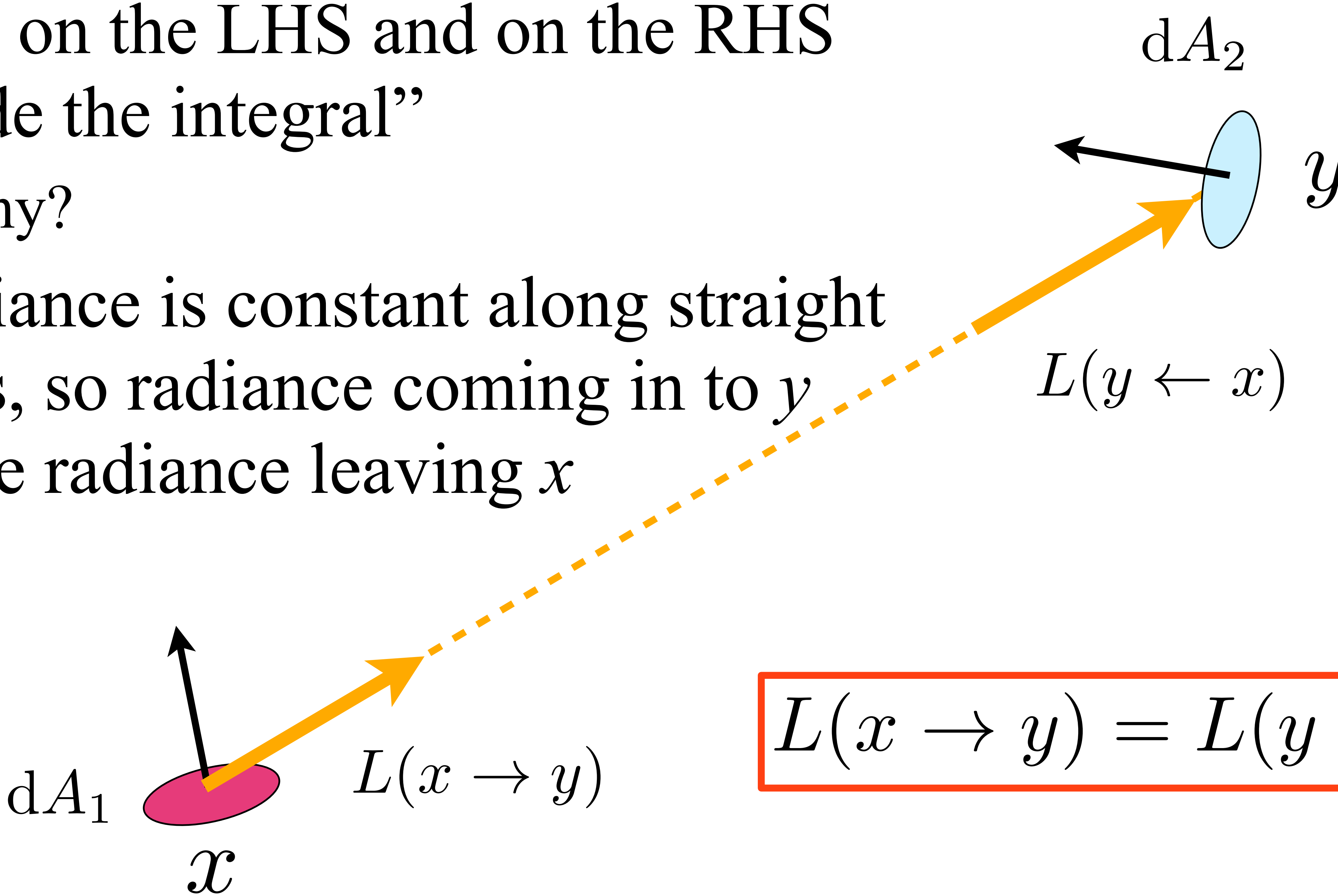
The Rendering Equation

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l}) \, f_r(x, \mathbf{l} \to \mathbf{v}) \, \cos\theta \, \mathrm{d}\mathbf{l} + E(x \to \mathbf{v})$$

- The rendering equation describes the appearance of the scene, including direct and indirect illumination
  - An "integral equation", the unknown solution function L is both on the LHS (left-hand side) and on the RHS inside the integral

Hmmh..

- “the unknown solution function L is both on the LHS and on the RHS inside the integral”
  - Why?
- Radiance is constant along straight lines, so radiance coming in to *y* is the radiance leaving *x*



$$\boxed{L(x \rightarrow y) = L(y \leftarrow x)}$$

The Rendering Equation

$$L(x \rightarrow \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v}) \,\cos\theta \,\mathrm{d}\mathbf{l} + E(x \rightarrow \mathbf{v})$$

- The rendering equation describes the appearance of the scene, including direct and indirect illumination
  - An "integral equation", the unknown solution function L is both on the LHS and on the RHS inside the integral
    - More precisely: a "Fredholm equation of the 2nd kind"
  - Originally described by Kajiya and Immel et al. in 1986
  - **Take a class in Functional Analysis to learn more!**

The Rendering Equation

- The unknown in this equation is the *function* $L(x \to \mathbf{v})$ defined for all points *x* and all directions **v**
- Analytic (exact) solution is impossible in all cases of practical interest
- Lots of ways to solve approximately
  - Monte Carlo techniques use random samples
  - Finite element methods (FEM) discretize using *basis functions*
    - Radiosity, wavelets, precomputed radiance transfer, etc.
  - Topic of next lecture!

Questions?

Stack Studios, Rendered using Maxwell

Next: Let's Distill Things Down

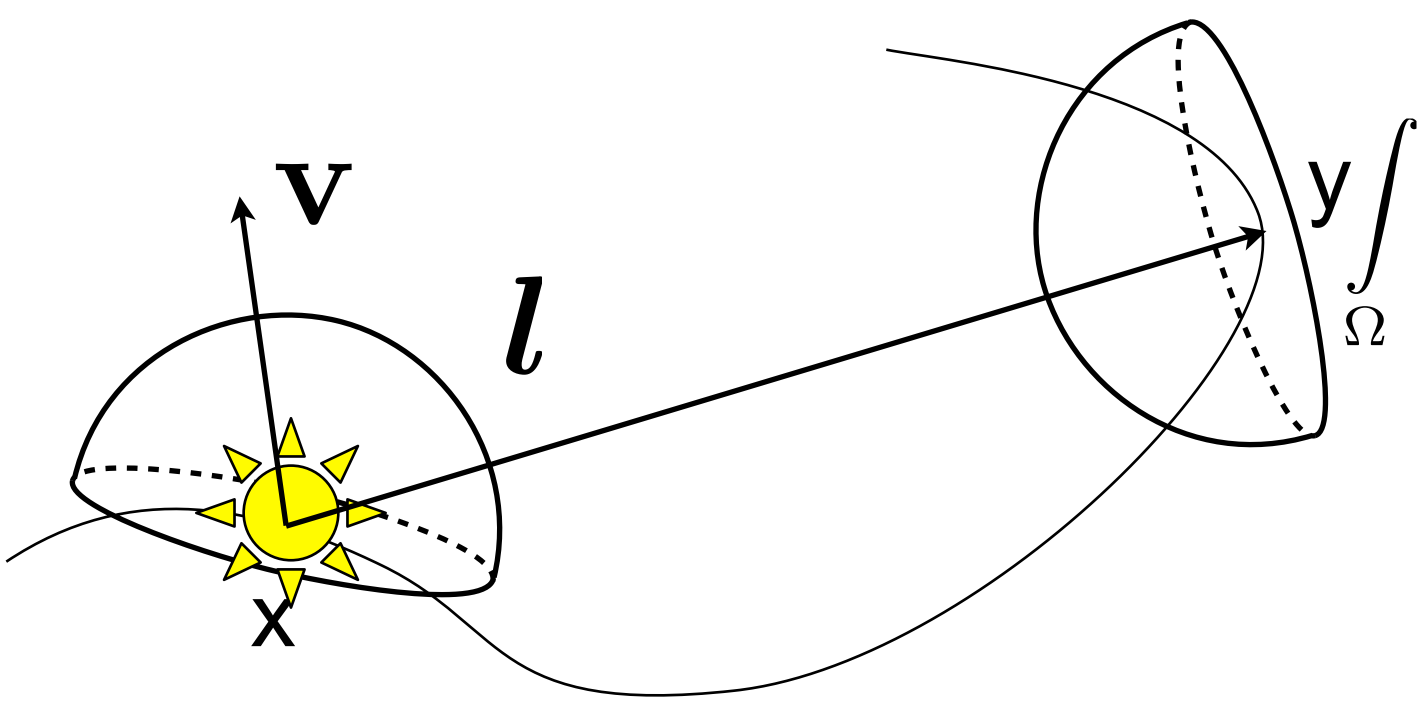
- We'll abstract the integrals away to see what is really going on: a linear system of equations
  - Infinitely many of them, though
  - Result: the *operator form* of the rendering equation

- This is helpful for..
  - understanding the nature of the solutions
  - thinking of numerical methods for solution

The Rendering Equation

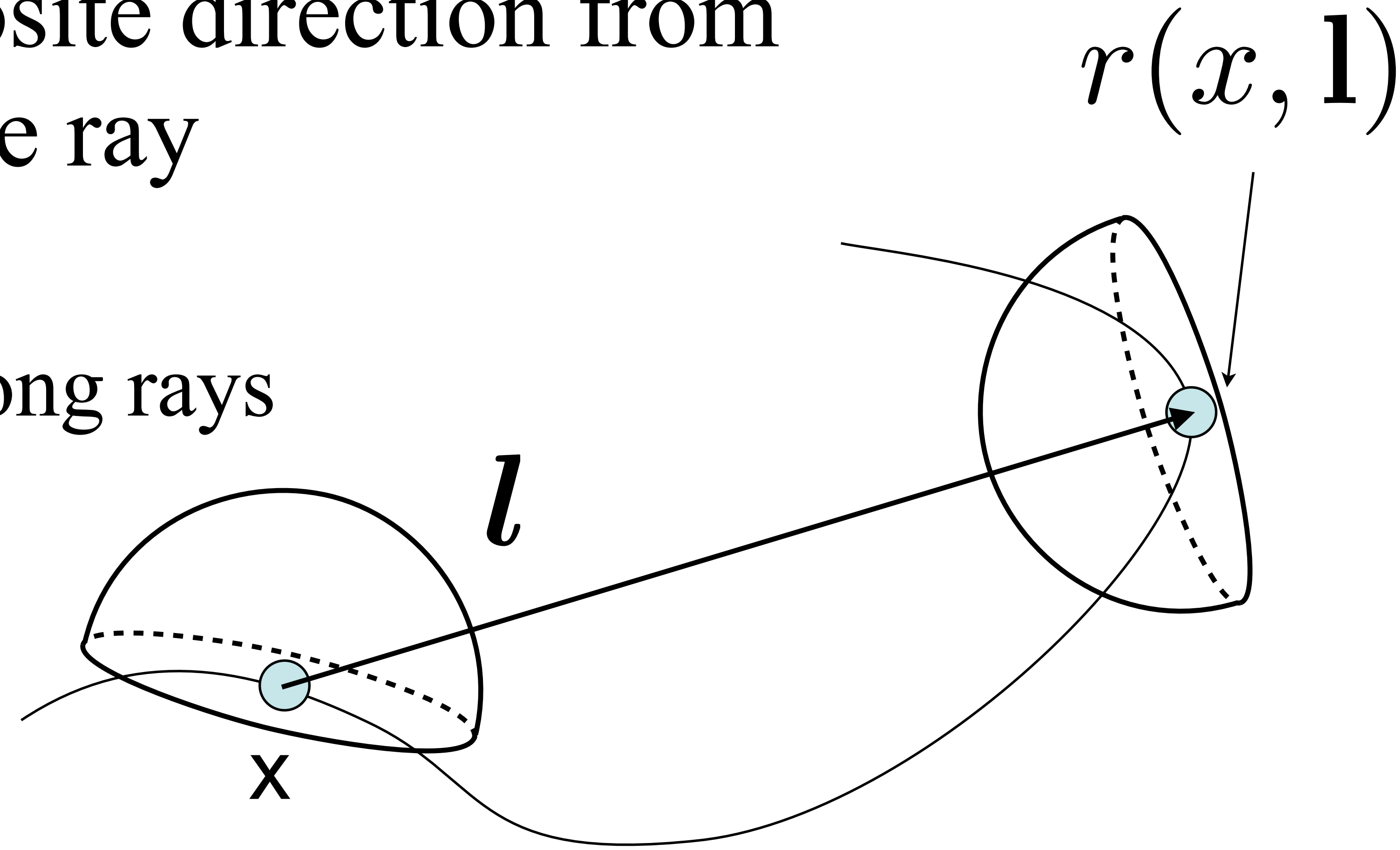
$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l} + E(x \to \mathbf{v})$$

- Recursive!
  - To know incident radiance at *x*, must know outgoing radiance at *all points y seen by x*



Operator Formulation 1

- "The lighting incident to x from $\mathbf{l}$ is the light exiting to the opposite direction from the point r(x,$\mathbf{l}$) where the ray from x towards $\mathbf{l}$ hits"
  - Constancy of radiance along rays
  - "Ray-cast function" $r(x,\mathbf{l})$ returns point hit by ray from x towards $\mathbf{l}$

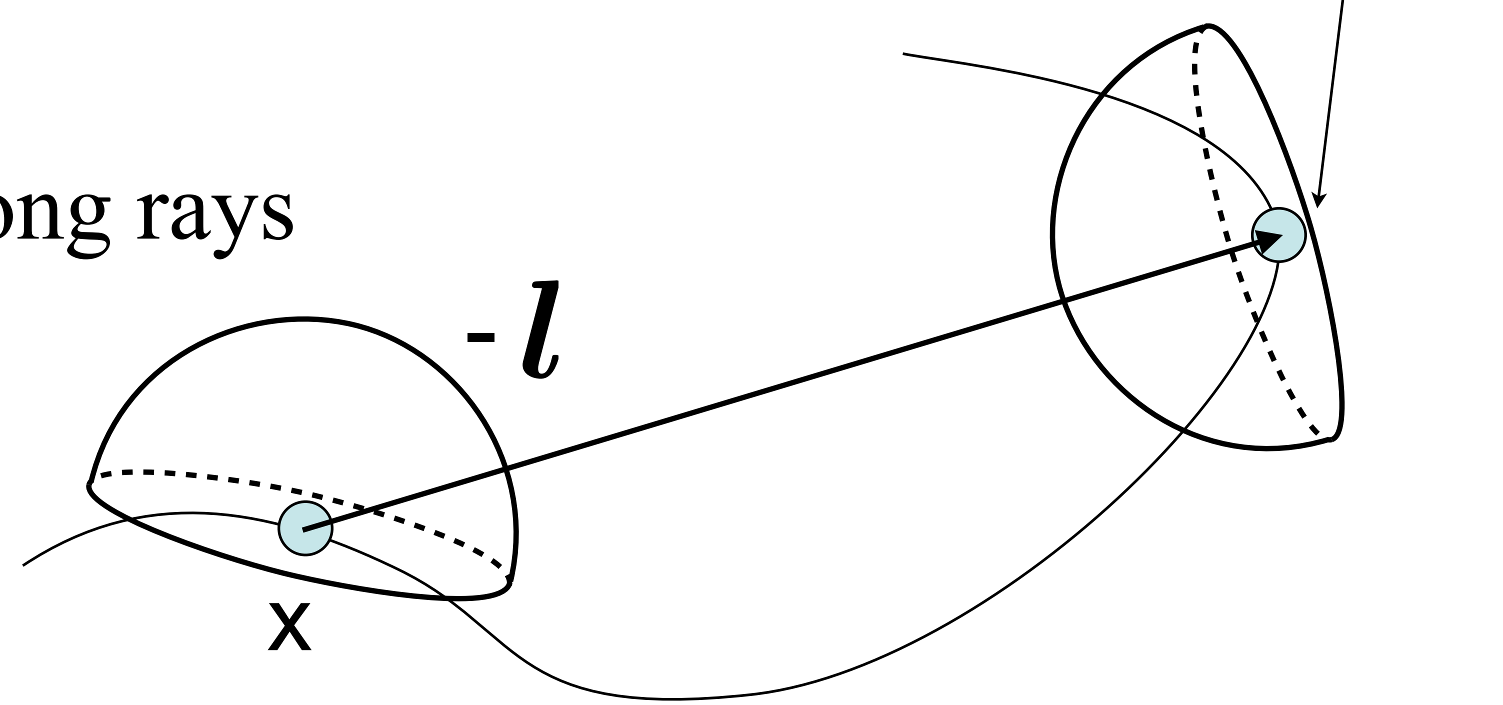


Operator Formulation 1

- Let's define the *propagation operator* G

$$L_{\text{in}}(x, \mathbf{l}) = (\mathcal{G}L_{\text{out}}) \stackrel{\text{def}}{=} L_{\text{out}}(r(x, \mathbf{l}) \to -\mathbf{l})$$

- "The lighting incident to x from $\mathbf{l}$ is the light exiting to the opposite direction from the point r(x,$\mathbf{l}$) where the ray from x towards $\mathbf{l}$ hits"
  - Constancy of radiance along rays
  - "Ray-cast function" $r(x,\mathbf{l})$ returns point hit by ray from x towards $\mathbf{l}$

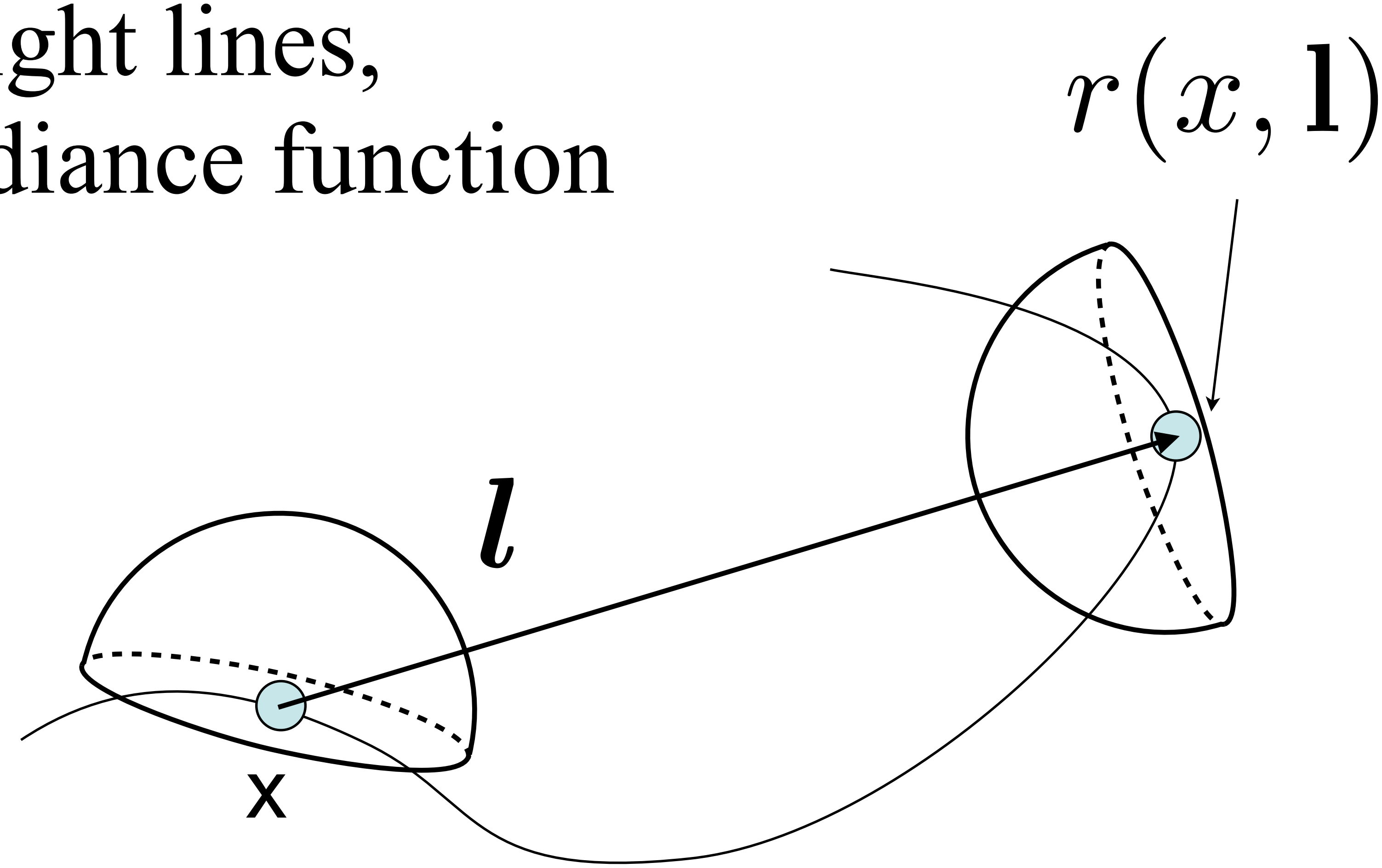


Operator Formulation 1

- Let's define the *propagation operator* G

$$L_{\mathrm{in}}(x, \mathbf{l}) = (\mathcal{G} L_{\mathrm{out}}) \stackrel{\mathrm{def}}{=} L_{\mathrm{out}}(r(x, \mathbf{l}) \rightarrow -\mathbf{l})$$

- G takes an outgoing radiance function, propagates it along straight lines, produces an incident radiance function



Operator Formulation cont'd

- ..and the *local reflection operator* R

$$L_{\text{out}}(x,\mathbf{v}) = (\mathcal{R}L_{\text{in}}) =$$

$$\int_\Omega L_{\text{in}}(x,\mathbf{l})\, f_r(x,\mathbf{l}\to\mathbf{v})\, \cos\theta\, \mathrm{d}\mathbf{l}$$

- Takes incident radiance function (defined for all points and directions), produces outgoing radiance function (defined for all points and directions)
- This is just another way of writing the reflectance integral you saw already

These operators are not complicated

take in one function
do something to it
return another function

runes.nu, rendered using Maxwell

Operator Form of Rendering Eq.

$$L_{\mathrm{out}}(x,\mathbf{v}) = \int_\Omega L_{\mathrm{in}}(x,\mathbf{l})\, f_r(x,\mathbf{l}\rightarrow\mathbf{v})\ \cos\theta_{\mathrm{in}}\,\mathrm{d}\mathbf{l} \\ + E_{\mathrm{out}}(x,\mathbf{v})$$

- Propagation + reflectance operators

$$L_{\mathrm{out}} = \mathcal{R} L_{\mathrm{in}}$$

$$L_{\mathrm{in}} = \mathcal{G} L_{\mathrm{out}}$$

Operator Form of Rendering Eq.

$$L_{\text{out}}(x, \mathbf{v}) = \int_{\Omega} L_{\text{in}}(x, \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta_{\text{in}}\, \mathrm{d}\mathbf{l} + E_{\text{out}}(x, \mathbf{v})$$

- Propagation + reflectance operators

$$L_{\text{out}} = \mathcal{R} L_{\text{in}}$$

$$L_{\text{in}} = \mathcal{G} L_{\text{out}}$$

- Let's put them together and add emission *E*:

$$\boxed{L_{\text{out}} = \mathcal{R}\mathcal{G} L_{\text{out}} + E}$$

Operator Form of Rendering Eq.

$$L_{\text{out}} = \mathcal{R}\mathcal{G}L_{\text{out}} + E$$

- Let's call propagation followed by reflection, the *transport operator T*: $\mathcal{T} = \mathcal{R}\mathcal{G}$
- Looks a lot like a linear system Ax=b, doesn't it?
  - Well, it *is* a linear system.
    Just with functions instead of vectors.
  - Easy to verify linearity: T(aX+bY) = aTX + bTY for any functions X,Y and scalars a,b

$$\boxed{(\mathcal{I} - \mathcal{T})L_{\text{out}} = E}$$

Consequence of Linearity

$$(\mathcal{I} - \mathcal{T}) L_{\text{out}} = E$$

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1}\, E$$

- This is kind of a deep result, although simple:
  the lighting solution is a linear function of the emission

Consequence of Linearity, Pt 2

$$(\mathcal{I} - \mathcal{T}) L_{\text{out}} = E$$

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} E$$

- Light is additive, i.e., we can break emission into parts

$$L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} (E_1 + E_2)$$

$$= \boxed{(\mathcal{I} - \mathcal{T})^{-1} E_1} + \boxed{(\mathcal{I} - \mathcal{T})^{-1} E_2}$$

“Neumann Series”

See link by clicking title!

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} E$$

- The Neumann series says

$$(\mathcal{I} - \mathcal{T})^{-1} = \sum_{i=0}^{\infty} \mathcal{T}^i$$

$$= \mathcal{I} + \mathcal{T} + \mathcal{T}^2 + \mathcal{T}^3 + \ldots$$

- I.e. $\boxed{L_{\text{out}} = E + \mathcal{T}E + \mathcal{T}\mathcal{T}E + \ldots}$

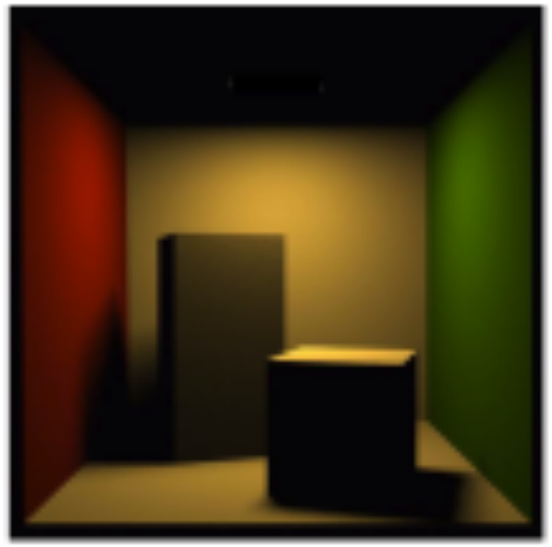
Neumann Series

$$\boxed{L_{\text{out}} = E + \mathcal{T}E + \mathcal{T}\mathcal{T}E + \ldots}$$

- The lighting solution is the sum of
  - emitted light E,
  - light reflected once TE,
  - light reflected twice TTE, etc.

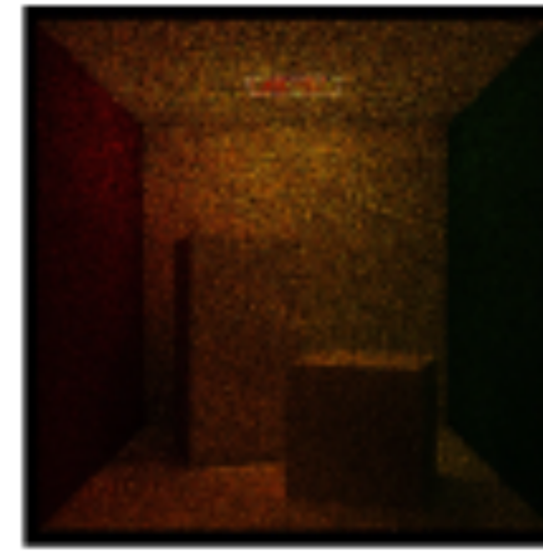
- Monte Carlo methods compute these integrals probabilistically

E

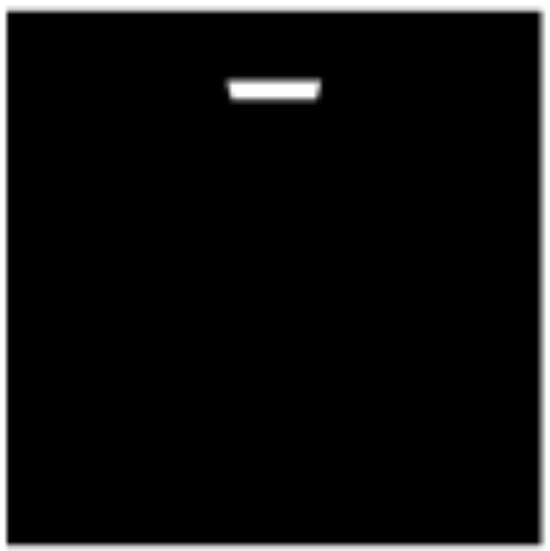


$\mathcal{T}E$

$\mathcal{T}\mathcal{T}E$

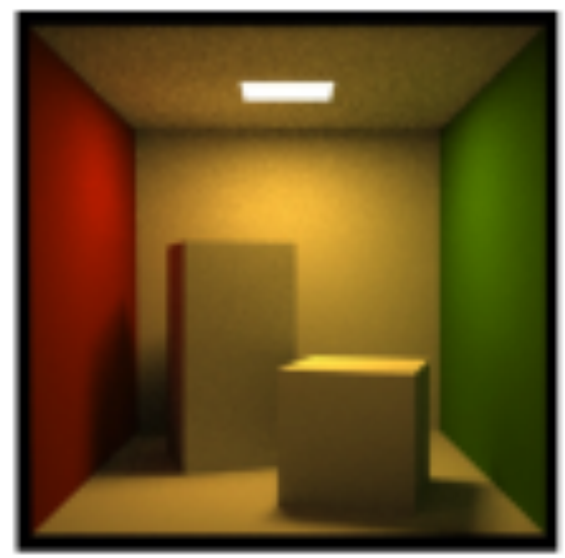


$\mathcal{T}\mathcal{T}\mathcal{T}E$

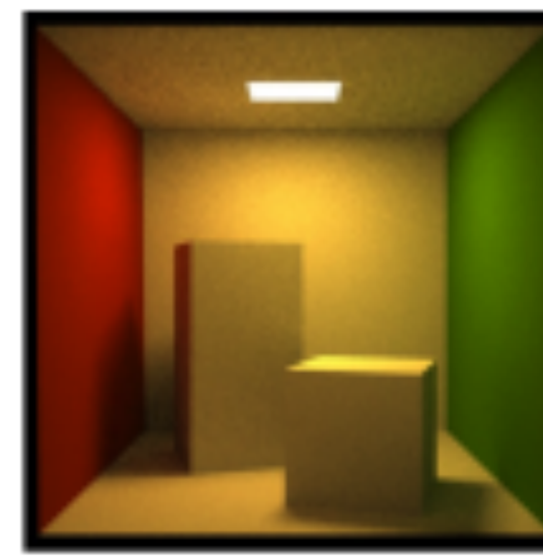


E

$E + \mathcal{T}E$



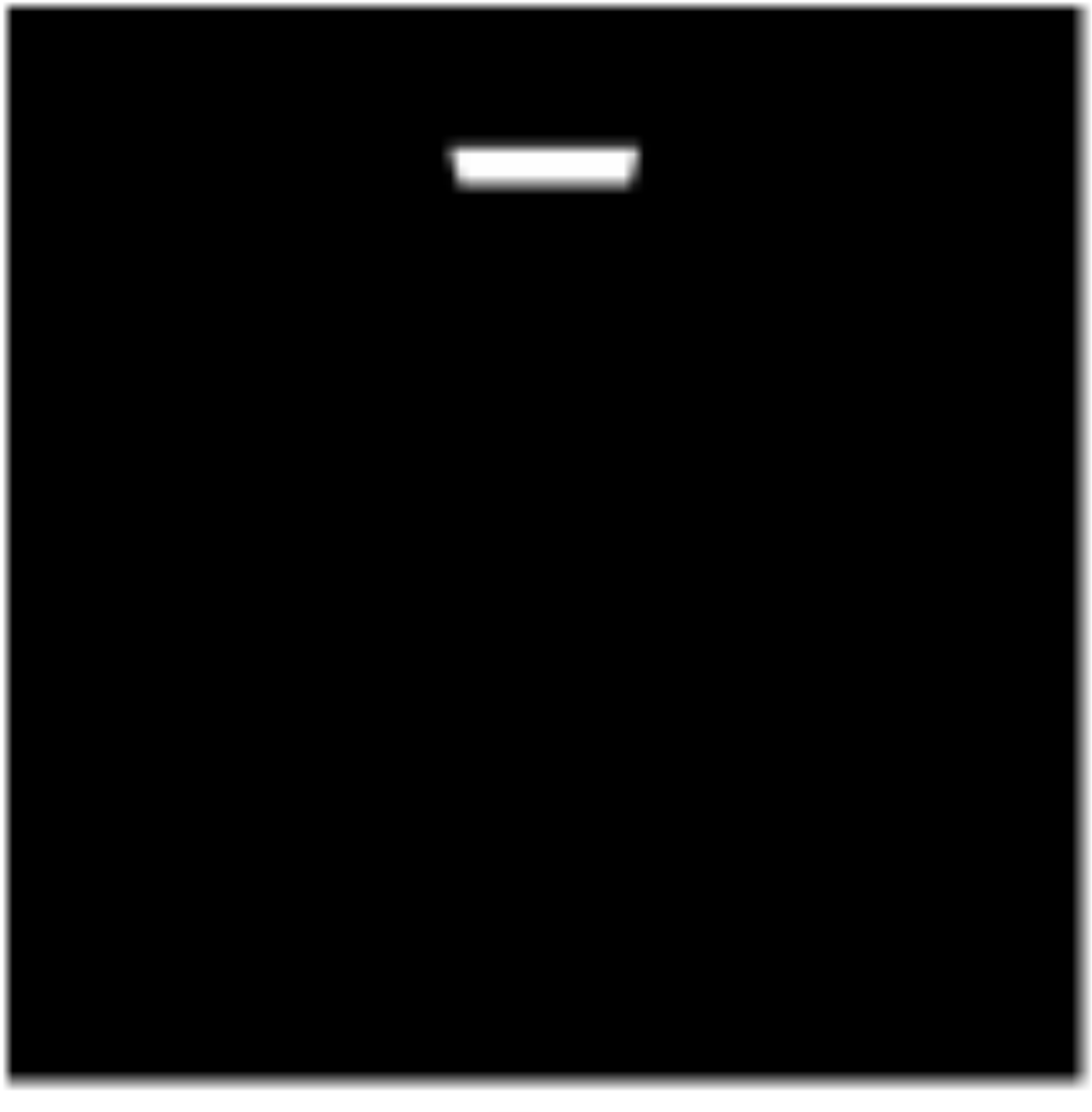
$$E + \mathcal{T}E + \mathcal{T}^2 E$$



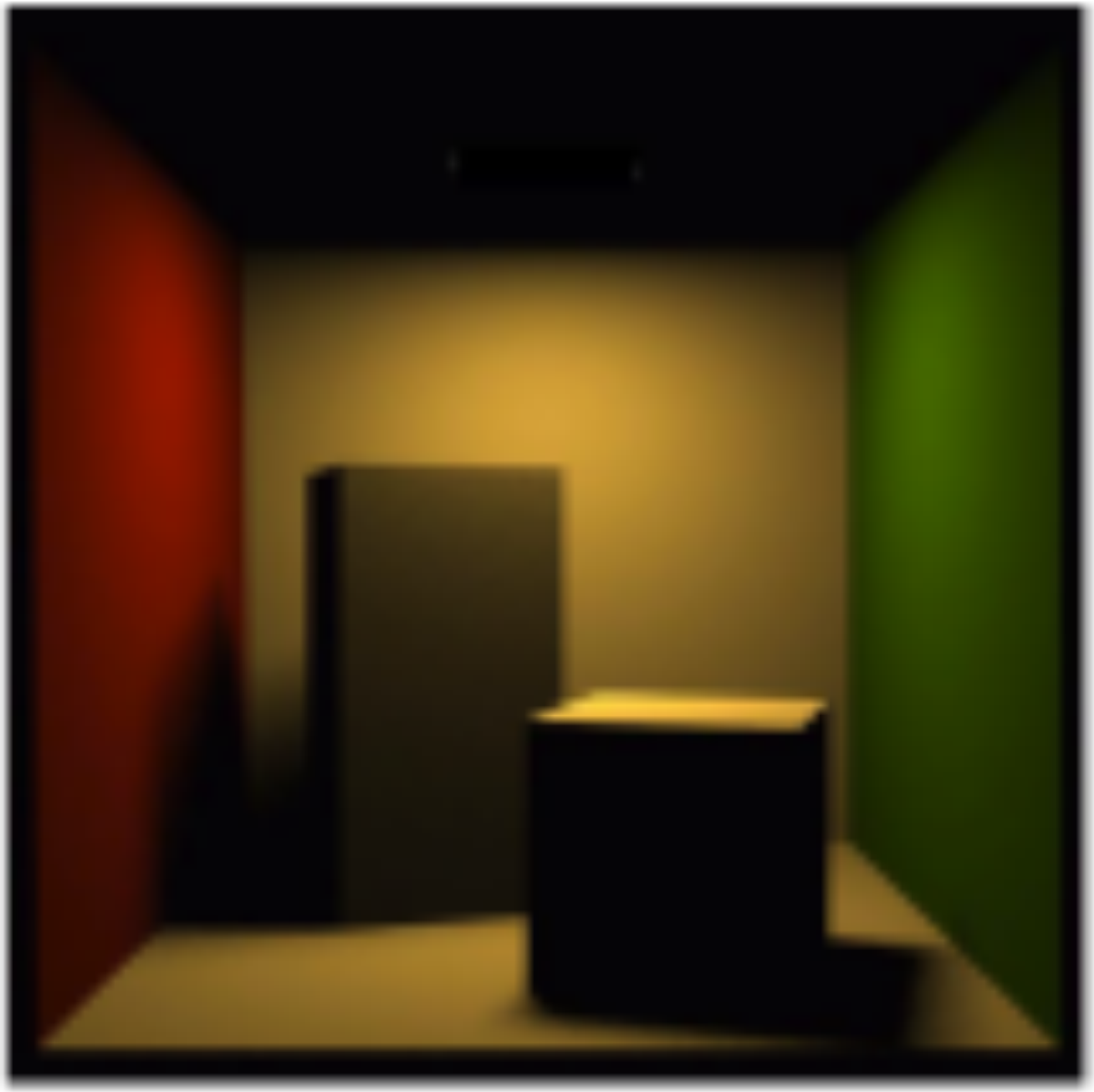
$$E + \mathcal{T}E + \mathcal{T}^2 E + \mathcal{T}^3 E$$

adapted from **Pat Hanrahan, Spring 2010**

E = Emitted Radiance (Light sources)

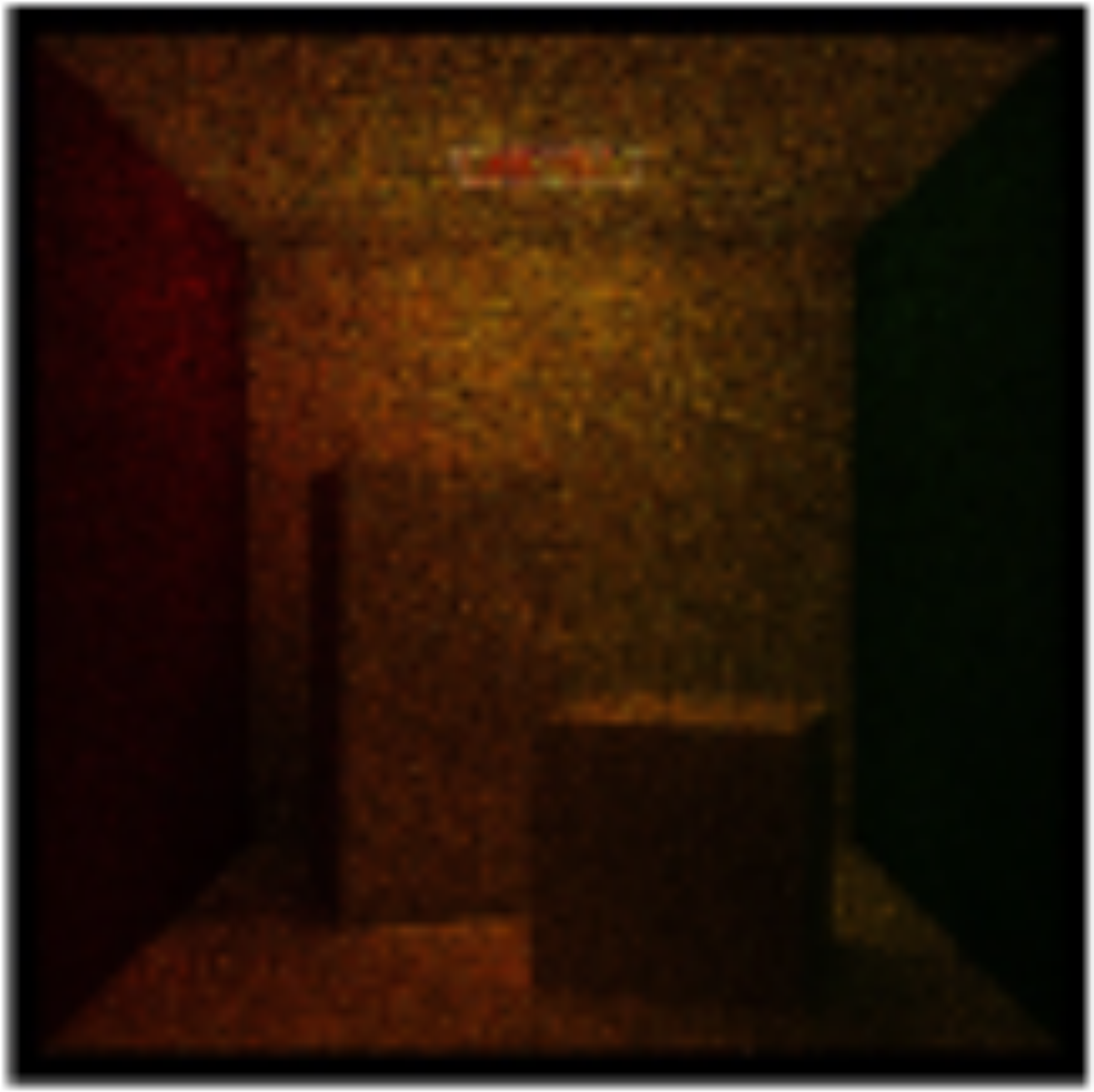


TE = Direct Lighting



TTE = First Indirect Bounce

TTTE = Second Indirect Bounce



Questions?