Lecture IV - Matching

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January 29, 2024

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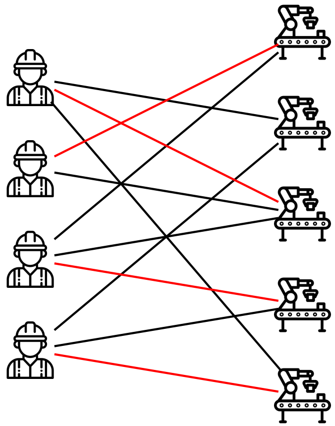
Previously on..

PREVIOUSLY ON...

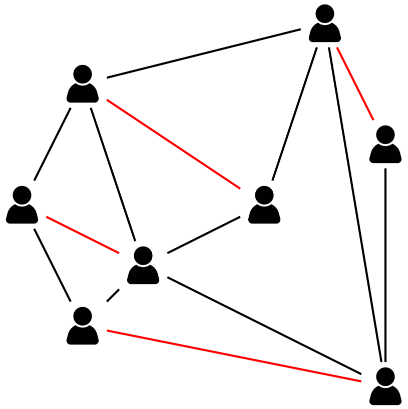
- Flow: Ford-Fulkerson;
- Flow: Edmond-Karp.

Both algorithms allow us to find what is the **maximum flow** that can be pushed through a **flow network**.

Motivation



assigning jobs to workers

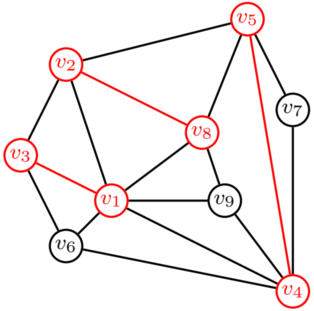


finding pairs for homework assignment

Definition

- $G = (V, E)$ undirected graph
- $M \subset E$ is called *matching* if all $e \in M$ are pairwise disjoint, i.e., if the endpoints are all different
- $M \subset E$ is a *maximum matching* in G if M is a matching with highest cardinality, i.e.,

$$|M'| \leq |M| \quad \text{for all matchings } M'$$



IP-formulation

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \leq 1, && v \in V \\ & x_e \in \{0,1\}, && e \in E \end{aligned}$$

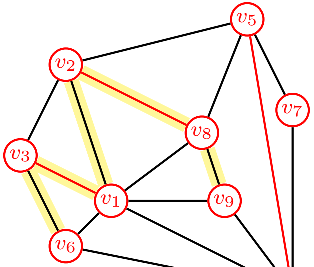
- incident edges of $v \in V$:

$$\delta(v) = \{e \in E \colon e = \{v, w\}\}$$

M-augmenting paths

- $G = (V, E)$ undirected graph, $M \subseteq E$ matching
- $v \in V$ is *covered* by M if $v \in e$ for some $e \in M$
- $v \in V$ is *exposed* by M if $v \notin e$ for all $e \in M$
- *M-alternating path P*: edges $E(P)$ are alternately in M and not in M (or not in M and in M)
- *M-augmenting path P*: M-alternating path with first and last vertex exposed

☙ M-augmenting paths have odd number of edges



Berge's Theorem

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Theorem (Petersen (1891), Berge (1957))

Let G be a graph with some matching M. Then M is maximum if and only if there is no M-augmenting path.

Proof idea $\Rightarrow$:

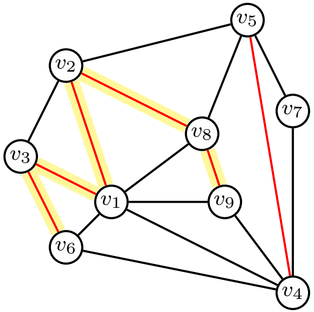
By contraposition: Let $P = (v_0, e_1, \dots, e_k, v_k)$ be an M-augmenting path.

- by definition: v_0, v_k exposed

$\Rightarrow$ $|E(P) \setminus M| = |E(P) \cap M| + 1$

$\Rightarrow$ $M' = (M \setminus E(P)) \cup (E(P) \setminus M)$ is matching with $|M'| = |M| + 1$

$\Rightarrow$ M not maximum



Lemma

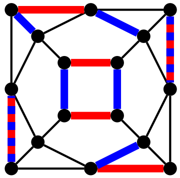
Lemma

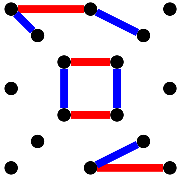
Let G be a graph with two matchings M, M'. Let $G' = (V, E' = M \Delta M')$, with symmetric difference

$$M \Delta M' = (M \cup M') \setminus (M \cap M').$$

Then the connected components of G' are

- *isolated vertices*
- *cycles C with $|E(C)| \in 2\mathbb{N}$ where edges in C are alternately in M and M'*
- *paths $P = (v_0, e_1, \dots, e_k, v_k)$ where edges are alternately in M and M'*


graph G


graph G'

Berge's Theorem

Proof idea:
M, M' matchings:

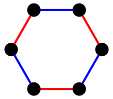
$$\begin{aligned} &|\{e \in M : v \in e\}| \leq 1, v \in V \\ &|\{e \in M' : v \in e\}| \leq 1, v \in V \\ \Rightarrow &|\{e \in E' : v \in e\}| \leq 2, v \in V \end{aligned}$$

If $g_{G'}(v) = |\{e \in E' : v \in e\}| = 2$:
$\exists! e \in M : v \in e$ and $\exists! e \in M' : v \in e$.

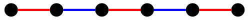
Berge's Theorem

- isolated vertices $v \rightsquigarrow g_{G'}(v) = 0$

- cycles C with $|E(C)| \in 2\mathbb{N} \rightsquigarrow g_{G'}(v) = 2$



- paths $P = (v_0, e_1, \ldots, e_k, v_k) \rightsquigarrow g_{G'}(v_0) = 0 = g_{G'}(v_k) = 1,\ g_{G'}(v_i) = 2,$ $1 \leq i \leq k-1$



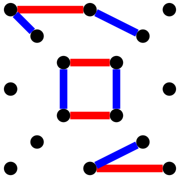
Berge's Theorem

Theorem (Petersen (1891), Berge (1957))

Let G be a graph with some matching M. Then M is maximum if and only if there is no M-augmenting path.

Proof idea $\Leftarrow$:

By contraposition: Let M' be a matching with $|M'| > |M|$. Construct G'.


graph G'

$$|M'| > |M| \Rightarrow |E' \cap M'| > |E' \cap M|$$
$$\Rightarrow \exists P = (v_0, e_1, \dots, e_k, v_k) \text{ with } e_1 \in M', e_k \in M'$$
$$\Rightarrow v_0, v_k \text{ exposed by } M$$
$$\Rightarrow P \ M\text{-augmenting path}$$

General algorithm

```
Algorithm: MAXIMUM MATCHING
Input: undirected graph $G = (V, E)$
Output: maximum matching $M$
1 set $M = \emptyset$
2 while there exists $M$-augmenting path in $G$ do
3     choose $M$-augmenting path $P$
4     set $M = (M \setminus E(P)) \cup (E(P) \setminus M)$
5 return $M$
```

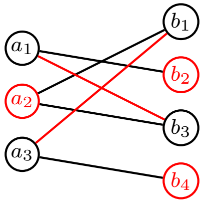
- at most $\frac{|V|}{2}$ iterations
- not obvious how to find an M-augmenting path
- for bipartite graphs: find s-t-path in auxiliary graph
- for general graphs: Edmond's blossom algorithm
  - complex algorithm
  - polynomial runtime

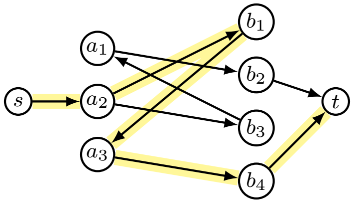
How to find M-alternating paths

- bipartite graph $G = (V, E)$ with
  - $V = A \cup B$, $A \cap B = \emptyset$
  - $E \subseteq \{\{a, b\} \colon a \in A, b \in B\}$
- for matching M construct auxiliary directed graph $G' = (V', E')$ with

$$\begin{aligned} V' =& V \cup \{s, t\}, \quad s, t \notin V \\ E' =& \{(b, a) \colon \{a, b\} \in M, a \in A, b \in B\} \\ & \cup \{(a, b) \colon \{a, b\} \in E \setminus M, a \in A, b \in B\} \\ & \cup \{(s, a) \colon a \text{ exposed}, a \in A\} \\ & \cup \{(b, t) \colon b \text{ exposed}, b \in B\} \end{aligned}$$

- $\exists$ M-augmenting path in G if and only if $\exists$ s-t-path in G'





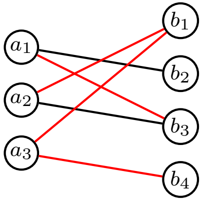
Maximum matching bipartite graphs – Example

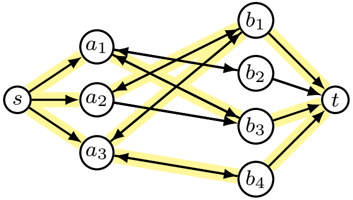
Algorithm: MAXIMUM MATCHING BIPARTITE GRAPHS

Input: undirected bipartite graph $G = (V, E)$

Output: maximum matching M

```
1 set $M = \emptyset$
2 construct $G'$
3 while there exists $s$-$t$-path in $G'$ do
4     choose $s$-$t$-path $P$
5     set $M = (M \setminus E(P)) \cup (E(P) \setminus M)$
6     update $G'$
7 return $M$
```





Maximum matching bipartite graphs – Analysis

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Algorithm: MAXIMUM MATCHING BIPARTITE GRAPHS
Input: undirected bipartite graph $G = (V, E)$
Output: maximum matching $M$
1 set $M = \emptyset$
2 construct $G'$
3 while there exists $s$-$t$-path in $G'$ do
4     choose $s$-$t$-path $P$
5     set $M = (M \setminus E(P)) \cup (E(P) \setminus M)$
6     update $G'$
7 return $M$
```

- $n = |V|, m = |E|$, no isolated nodes in G
- construction of G': $O(n + m)$
- at most $\frac{n}{2}$ iterations
  - finding P: $O(m)$
  - updating M: $O(n)$
  - updating G': $O(n)$

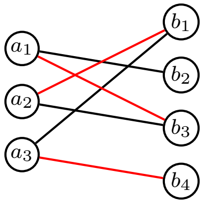
$\Rightarrow$ total runtime $O(nm)$

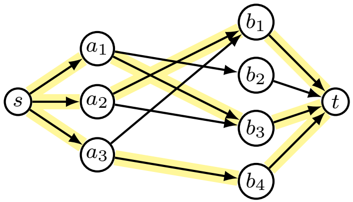
Connection to MaxFlow

- solve maximum matching as maximum flow problem
- construct auxiliary directed graph $G'' = (V'', E'')$ with

$$\begin{aligned} V'' =& V \cup \{s,t\}, \quad s,t \notin V \\ E'' =& \{(a,b) \colon \{a,b\} \in E, a \in A, b \in B\} \\ & \cup \{(s,a) \colon a \in A\} \\ & \cup \{(b,t) \colon b \in B\} \end{aligned}$$

  and capacity $u(e) = 1$ for all $e \in E''$
- G'' has maximal flow with value k if and only if G has maximum matching of cardinality k





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