K1

$$\begin{cases} x = e^s \cos t \\ y = e^s \sin t \end{cases} \Rightarrow \begin{cases} \dfrac{\partial x}{\partial s} = x, & \dfrac{\partial x}{\partial t} = -y \\ \dfrac{\partial y}{\partial s} = y, & \dfrac{\partial y}{\partial t} = x \end{cases}$$

Käytetään osittaisderivaatan ketjusääntöä:

$$\frac{\partial z}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial^2 z}{\partial s} = \frac{\partial}{\partial s}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] = \left(\frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y}\right)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

$$= x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

ja vastaavasti

$$\frac{\partial z}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial z}{\partial y} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = \left(\frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y}\right)\left[-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right]$$

$$= y^2\frac{\partial^2 z}{\partial x^2} - y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} + x^2\frac{\partial^2 z}{\partial y^2}$$

josta seuraa

$$\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = (x^2 + y^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

K2 $\quad u(x,y) = r^2 \ln(r) \quad , \quad r = \sqrt{x^2+y^2}$

Huomataan, että $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$.

Käytetään osittaisderivaatan ketjusääntöä :

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} = \frac{x}{r}\left(2r\ln(r) + \frac{r^2}{r}\right) = x(1+2\ln(r))$$

$$\frac{\partial^2 u}{\partial x^2} = (1+2\ln(r)) + \frac{2x^2}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} = y(1+2\ln(r)) \Rightarrow \frac{\partial^2 u}{\partial y^2} = (1+2\ln(r)) + \frac{2y^2}{r^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 4\ln(r) + \frac{2(x^2+y^2)}{r^2} = \underline{4 + 4\ln(r)}$$

eli huomataan että $u(x,y)$ ei ole harmoninen.

Jatketaan (käyttäen edelleen ketjusääntöä) :

$$\frac{\partial}{\partial x}\left[4+4\ln(r)\right] = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2}$$

$$\frac{\partial^2}{\partial x^2}\left[4+4\ln(r)\right] = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{1}{r^2} + x\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) = \frac{1}{r^2} - \frac{2x^2}{r^4}$$

ja vastaavasti

$$\frac{\partial}{\partial y}\left[4+4\ln(r)\right] = \frac{y}{r^2}$$

$$\frac{\partial^2}{\partial y^2}\left[4+4\ln(r)\right] = \frac{1}{r^2} - \frac{2y^2}{r^4}$$

Todetaan

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[4+4\ln(r)\right] = \frac{2}{r^2} - \frac{2(x^2+y^2)}{r^4} = 0$$

eli $u(x,y)$ on biharmoninen !