§ Week V §

Problem 1: TSP

Consider the Travelling Salesman Problem (TSP) where a salesman must visit each of n given cities $V = \{1, \cdots, n\}$ exactly once and then return to his starting point. The distance between two cities i and j is given by c_{ij}. The goal is to determine a tour of minimum length. We following ILP is proposed to solve this problem:

$$\textit{Minimize} \quad \sum_{i \in \{1,\ldots,n\}} \sum_{j \in \{1,\ldots,n\}} c_{ij} x_{ij} \tag{1.1a}$$

Subject to:

$$\sum_{j:j \neq i} x_{ij} = 1 \qquad \forall i \in \{1, \ldots, n\} \tag{1.1b}$$

$$\sum_{i:i \neq j} x_{ij} = 1 \qquad \forall j \in \{1, \ldots, n\} \tag{1.1c}$$

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} x_{ij} \leq |\mathcal{S}| - 1 \qquad \forall \emptyset \neq \mathcal{S} \subset \mathcal{V} \tag{1.1d}$$

$$x \in \{0, 1\} \tag{1.1e}$$

The goal is to show that this is a correct formulation of the TSP.

1. Give an interpretation of the binary variables x_{ij} and the constraints in the above program;

   Solution:
   For each edge in E, a binary variable is associated to it. If an edge is used as part of a solution, the binary value assumes the value of 1.

   Constraints 1.1b and 1.1c impose that each node has exactly one incoming edge and one outgoing edge.

   Constraint 1.1d imposes that for any subset $S \subset V$, there is, at most, $|S| - 1$ edges selected.

2. Prove that this formulation is correct. Start by showing that without the subtour elimination constraints every feasible solution to the above IP consists of vertex-disjoint cycles.

   Solution:
   The following solutions below satisfy constraint 1.1b and 1.1c. However, constraint 1.1d is only satisfied the solution on the right-hand side.

3. The following set of constraints are called the cut-set constraints for the TSP:

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{V} \setminus \mathcal{S}} x_{ij} \geq 1 \qquad \forall \emptyset \subseteq \mathcal{S} \subset \mathcal{V} \tag{1.2}$$

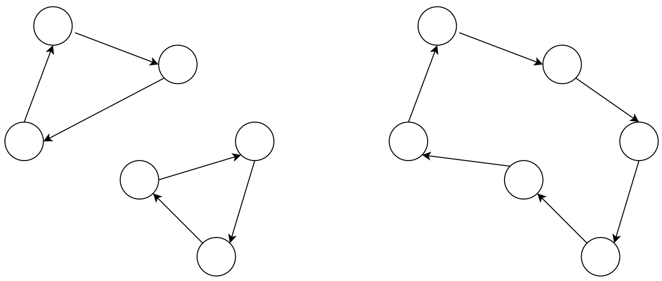

Figure 1: Both solutions satisfy the first two constraints, but only the solution in the right-hand side satisfy subtour elimination

Show that replacing the subtour elimination constraints by the cut-set constraints yields an alternative valid formulation of the TSP.

Solution:

The additional constraint in 1.2 imposes that for any subset $S \subseteq V$, at least one edge has to be connecting to a node outside of the subset. With this, it indirectly eliminates the chance of subtour inside each subset. See example below:

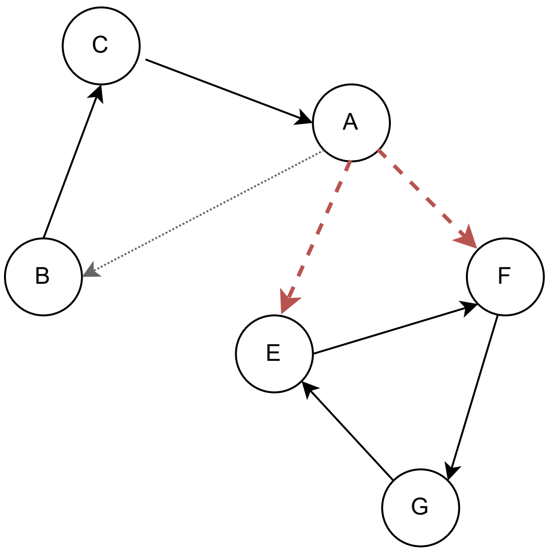

Figure 2: The constraint forces that either x_{AE} or x_{AF} is selected (in red, dashed line), given that they are edges from outside the subset $\{A, B, C\}$. Due to the other constraints, x_{AB} (in grey, dotted line) cannot be selected then, thus breaking the subtour.

Problem 2: Graph Colouring

A k-colouring $k \in \mathcal{N}$ for an undirected graph $G = (V, E)$ is a surjective mapping $f : \mathcal{V} \to \{1, \cdots, k\}$ from the edges of G into the numbers $\{1, \cdots, k\}$ such that for each $(u, v) \in \mathcal{E}$ it holds $f(u) \neq f(v)$. The minimum graph colouring problem asks for the minimum k such that there exists a k-colouring in G. Give an IP-formulation for the minimum graph colouring problem and prove its correctness.

Solution:

For each node $v \in V$ and each colour $i \in \{1, \cdots, H\}$, we are introducing a binary variable x_{vi} such that:

$$x_{ij} = \left\{ \begin{array}{ll} 1, & \text{if node } v \text{ is assigned to colour } i \\ 0, & \text{otherwise} \end{array} \right\}$$

For each color $i \in \{1, \cdots, H\}$, we are also introducing w_i such that:

$$w_i = \left\{ \begin{array}{ll} 1, & \text{if colour } i \text{ is used} \\ 0, & \text{otherwise} \end{array} \right\}$$

The constraints are:

1. Every node has to be assigned to at least a colour.

$$\sum_{i=1}^{H} x_{vi} = 1 \qquad \forall v \in V \tag{2.1}$$

2. The colours of two neighbouring nodes have to be different.

$$x_{ui} + x_{vi} \leq w_i \qquad \forall (u,v) \in E, i = \{1, \cdots, H\} \tag{2.2}$$

The objective function is to minimize the amount of colours used:

$$\min \sum_{1 \leq i \leq H} w_i \tag{2.3}$$

Problem 3: Linear vs Non-Linear

In contrast to non-linear programs, integer programs as discussed in this course cannot contain products of variables. In some cases, however, it is possible to replace such products in an integer program at the cost of introducing new variables. Consider the non-linear constraint $x \cdot y \leq b$ with $b \in \mathcal{R}_0^+$, where:

1. $x, y \in \{0, 1\}$;
2. $x \in [0, \mathcal{M}], \mathcal{M} \in \mathcal{R}_0^+, y \in \{0, 1\}$ (where M is a large constant);

and find an equivalent linear formulation for each.

Solution:

In the first case, the goal is to linearise a product of two binary variables. This can be achieved by adding a new variable ϕ (also binary) such that:

$$\phi \leq x \tag{3.1}$$
$$\phi \leq y \tag{3.2}$$
$$\phi \geq x + y - 1 \tag{3.3}$$

In the second case, the goal is to linearise a product of a binary variable and a continuous variable. Once again, we introduce the variable ϕ (as continuous):

$$\phi \leq My \tag{3.4}$$
$$\phi \leq x \tag{3.5}$$
$$\phi \geq x - (1-y)M \tag{3.6}$$
$$\phi \geq 0 \tag{3.7}$$

Problem 4: Presidential Debate

Assume you are running for election in a country with five states $S = \{s_1, s_2, s_3, s_4, s_5\}$ and a two- party system. Each state $s \in S$ provides v_s votes (where the sum $\sum_s v_s$ is odd) and you need at least half of these votes to win. To obtain all v_s votes of the state s, you must win the popular vote in this state (or tie, tying is fine).

Currently, polls predict you will win a p_s fraction of the votes in state s while your political opponent is estimated to obtain a $\overline{p}_s$ fraction of them. Here $p_s \in [0, 1]$ and $\overline{p}_s \leq 1 - p_s$.

The election is imminent and your opponent is confident that they will win (which is an assumption made to justify them doing nothing to make the problem simpler). You, on the other hand, decide to use the remaining time to organise another eight campaign rallies. Because the election is right around the corner, you can only hold rallies in at most four of the five states and (strictly) more than three in the same state promise to be useless.

However, each of your rallies up to the third convinces a fraction f_s of the currently still undecided voters to vote for you. This means that your first rally in the state s would increase your percentage from p_s to $p_s + f_s(1 - p_s - \overline{p}_s)$. Note that this does have diminishing returns, the second rally is already only $(1 - f_s)$ as effective.

Use an IP to determine whether you can still win this election with the help of these rallies and where these would need to be. For the sake of this exercise, you may assume that the polls are entirely accurate (which is always the case, you have nothing to worry about).

Solution:

The following ILP solves this problem:

$$
\begin{aligned}
&\text{Maximize} \quad \sum_s v_s w_s \\
&\text{Subject to:} \\
&w_s \leq 1 + p_s + p_s^1 x_s^1 + p_s^2 x_s^2 + p_s^3 x_s^3 - \overline{p}_s && \forall s \in S \\
&\sum_{i,s} x_s^i = 8 \\
&x_s^1 \geq x_s^2 \geq x_s^3 && \forall s \in S \\
&x_s, w_s \in \{0, 1\} && \forall s \in S
\end{aligned}
$$

In this IP, the value p^i denotes the benefit of holding the i-th rally, that is, $p^i = f_s(1 - p_s - \overline{p}_s)(1 - f_s)^{i-1}$. Also the variable x_s^i represents whether the i-th rally takes place in state s. The IP now describes the maximum amount of votes that are obtainable by using 8 rallies.

First assume that we decide to hold i_s rallies in state s, which lets us win states in a set W. We obtain a feasible solution by setting x_s^i to 1 if $i \leq i_s$ and we set it to 0 otherwise. We also set w_s to 1 if $s \in W$ and to 0 otherwise. This solution is feasible. By construction it satisfies the second, third, and fourth constraint and the first one holds true as we obtain exactly a $p_s^* = p_s + p_s^1 x_s^1 + p_s^2 x_s^2 + p_s^3 x_s^3$ percentage of the votes in state s. This means that, in any won state, the RHS of this equation is at least 1. The objective then correctly describes the amount of votes obtained.

Conversely, a feasible solution x_s, w_s to the IP yields a rally plan by holding $\sum_i x_s^i$ rallies in state s. This plan is feasible: in each state there are at most three rallies, as we only have three such variables, and we only hold rallies in at most four states since the last two constraints guarantee that a positive amount of rallies is state s require x_s^i to be set to 1 and there are at most four such states.
This rally plan lets us win a state s if and only if $p_s^* \geq \overline{p}_s$ which is equivalent to the RHS of the first constraint being at least 1. Consequently, if w_s is set to 1, we also win that state and we obtain at least the amount of votes specified by the objective. In optimal solutions, we also set w_s to 1 if this is possible.