Lecture 11: The most common changes of variables: polar, cylider and spherical coordinates

Learning goals:

1. What are polar coordinates and how they can be used in the change of variables?
2. What are cylinder coordinates and how they can be used in the change of variables?
3. What are spherical coordinates and how they can be used in the change of variables?

Where to find the material?

Corral 3.5

Guichard et friends 15.2, 15.6

Active Calculus 11.5, 11.8

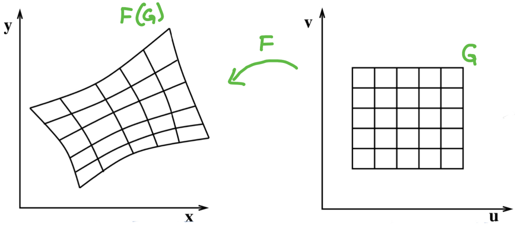
Adams-Essex 15.4, 15.6

Last time

The change of variables formula for double integrals

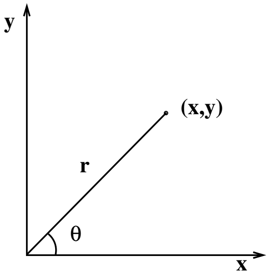
$$\iint_D f(x,y)\,dA = \iint_G g(u,v)\left|\det D\mathbf{F}(u,v)\right|\,du\,dv$$

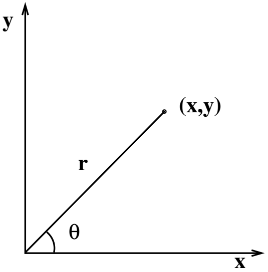
where $g(u,v) = f(x(u,v), y(u,v)) = f(\mathbf{F}(u,v))$ ja $D = \mathbf{F}(G)$.
And $\mathbf{F}(u,v) = (x(u,v), y(u,v))$ is bijection between G and D with continuous 1st order partial derivatives.



Polar coordinates

- Coordinate systems let's us to use algebraic methods to understand geometry
- A coordinates system is a scheme that allows us to identify any point in the plane or the space by a set of numbers.
- Rectangular coordinates (Cartesian coordinates) are most common, but sometimes using alternate coordinate systems makes problems easier.
- In **polar coordinates** a point $(x, y) \in \mathbb{R}^2$ can be written in a form (r, θ), where $r \geq 0$ and $0 \leq \theta < 2\pi$.





- Using geometry we get the following formulas

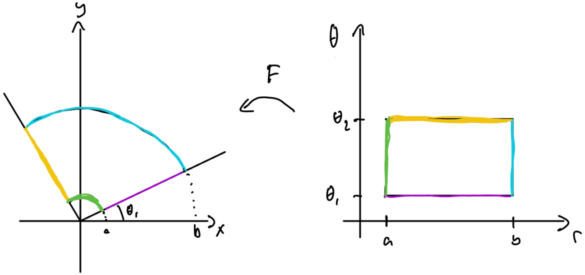
$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \Leftrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x. \end{cases}$$

Polar coordinates in the change of the variable formula

- The change of the variable formula:

$$\iint_D f(x,y)\,dA = \iint_G f(x(u,v),y(u,v))\left|\det D\mathbf{F}(u,v)\right| du\,dv$$

- The polar coordinate change



$$F(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$$

- Thus

$$\det D\mathbf{F}(r,\theta) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

- So the double integral in polar coordinates is

$$\iint_D f(x,y)\,dA = \iint_G g(r,\theta) r\,dr\,d\theta,$$

where $g(r,\theta) = f(r\cos\theta, r\sin\theta)$.

Example

- Let $D = \{(x, y) \in: 1 < x^2 + y^2 < 4\}$.
- Calculate the integral

$$I = \iint_D \frac{1}{x^2 + y^2}\, dx\, dy.$$

- The shape of the D (draw a picture) and the integrand suggest that this is easier to do in the polar coordinates
- For polar coordinates we have the formulas

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \Leftrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x. \end{cases}$$

- Thus

$$I = \int_0^{2\pi} \int_1^2 \frac{1}{r^2} r\, dr\, d\theta = \int_0^{2\pi} \int_1^2 \frac{dr}{r}\, d\theta$$

$$= 2\pi \ln r \Big|_{r=1}^{2} = 2\pi \ln 2.$$

Second example

Calculate

$$\iint_D \arctan^2(\frac{y}{x})\, dA,$$

where $D = \{(x, y) : 1 < x^2 + y^2 < 2 \text{ and } x, y > 0\}$.

(Answer: $\frac{\pi^3}{48}$)

What is the physical interpretation for the integral?

A famous example of the using the polar coordinates

- The integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

  is particularly important in, among other things, probability and statistics.
- The integral is difficult because it is not possible to write an integral function using elementary functions.
- However, it is possible to calculate the integral by the following trick:

$$I = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2.$$

We can calculate the improper double integral in polar coordinates:

$$I = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^{\infty} re^{-r^2} \, dr$$

$$= 2\pi \int_0^{\infty} re^{-r^2} \, dr = -\pi \lim_{R \to \infty} \int_0^R (-2r) e^{-r^2} \, dr$$

Because $\frac{d}{dr} e^{-r^2} = -2re^{-r^2}$ we have

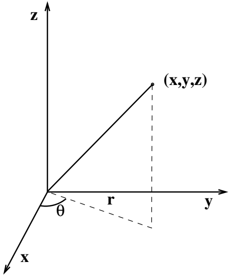
$$\int_0^R (-2r) e^{-r^2} \, dr = e^{-R^2} - 1$$

Letting $R \to \infty$ we get $I = \pi$ and thus the value of the original integral:

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{I} = \sqrt{\pi}.$$

Cylinder coordinates

- In **cylinder coordinates** a point $(x, y, z) \in \mathbb{R}^3$ can be given in a form (r, θ, z), where $r \geq 0$, $0 \leq \theta < 2\pi$, $z \in \mathbb{R}$.



From geometry we get the formulas:

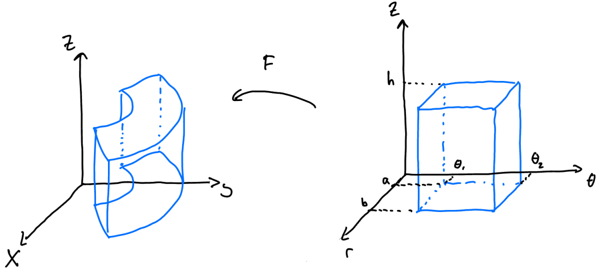
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

Cylinder coordinates in the change of the variable formula

- The change of the variable formula

$$\iiint_D f(x,y,z)\,dV =$$
$$\iiint_G f(x(u,v,w),y(u,v,w),z(u,v,w))\Big|\det D\mathbf{F}(u,v,w)\Big|\,du\,dv\,dw$$

- The cylinder coordinate change



$$F(r,\theta,z) = (x(r,\theta,z),y(r,\theta,z),z(r,\theta,z)) = (r\cos\theta, r\sin\theta, z)$$

- Thus

$$\det D\mathbf{F}(r,\theta,z) = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

- The triple integral in cylinder coordinates

$$\iiint_D f(x,y,z)\, dV = \iiint_G g(r,\theta,z) r\, dr\, d\theta\, dz,$$

where $g(r,\theta,z) = f(r\cos\theta, r\sin\theta, z)$.

Where to use cylinder coordinates?

Cylinder coordinates make it easy to handle rotations around z-axis, because a curve rotating around z-axis can be written in a form

$$r = f(z), \quad \text{where} \quad z \in [a, b] \text{ and } \theta \in [0, 2\pi),$$

where f is a non-negative function.

Example

Calculate the volume of the solid Ω of revolution of the area between the axis and f.

So

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \; : \; a \leq z \leq b, \; \sqrt{x^2 + y^2} \leq f(z)\}.$$

$$\iiint_\Omega dx\, dy\, dz = \int_a^b \int_0^{2\pi} \int_0^{f(z)} r\, dr\, d\theta\, dz$$

$$= \int_a^b \left(2\pi \cdot \frac{1}{2} f(z)^2\right) dz = \pi \int_a^b f(z)^2\, dz.$$

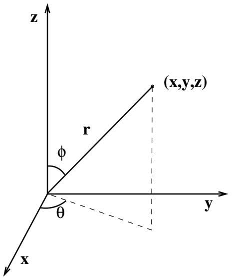
Second example

Find the volume under $z = \sqrt{4 - x^2 - y^2}$ above the quarter circle inside $x^2 + y^2 = 4$ in the first quadrant.

(Answer$= \frac{4\pi}{3}$)

Spherical coordinates

- In spherical coordinates a point (x, y, z) can be given in a form (r, θ, ϕ), where $r \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$.



- From geometry we get the formulas

$$\begin{cases} x = r \sin\phi \cos\theta \\ y = r \sin\phi \sin\theta \\ z = r \cos\phi \end{cases}$$

Spherical coordinates in the change of the variable formula

- Now $F(r,\theta,\phi) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$
- Thus

$$\begin{aligned}\det D\mathbf{F}(r,\theta,\phi) &= \det\begin{bmatrix}\sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta\\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta\\ \cos\phi & 0 & -r\sin\phi\end{bmatrix}\\ &= -r^2\sin\phi\end{aligned}$$

- Absolute value of this is $r^2\sin\phi$
- So the triple integral in spherical coordinates

$$\iiint_D f(x,y,z)\,dV = \iiint_G g(r,\theta,\phi)r^2\sin\phi\,dr\,d\theta\,d\phi,$$

where $g(r,\theta,z) = f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$.

Example

Calculate the volume of a ball $\mathbb{B}^3(R)$ of radius R:

$$\iiint_{\mathbb{B}^3(R)} 1\, dV = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\phi\, d\phi\, d\theta\, dr$$

$$= \int_0^R \int_0^{2\pi} -r^2 \cos\phi \Big|_{\phi=0}^{\pi} d\theta\, dr = \int_0^R \int_0^{2\pi} 2r^2\, d\theta\, dr$$

$$= \int_0^R 2 \cdot 2\pi r^2\, dr = \frac{4\pi r^3}{3}\Big|_{r=0}^{R} = \frac{4\pi R^3}{3}.$$