Lecture VII - NP Problems and Polynomial Transformation

[1] Department of Mathematics and Systems Analysis,
Systems Analysis Laboratory, Aalto University, Finland

March 4, 2024

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TSP and Hamiltonian Cycle

Definition

Both problem are related to find a **cycle**.

TSP and Hamiltonian Cycle reduction:

- For a graph $G = (V, E)$, build a complimentary graph G';
- For every pair of nodes (u, v) without an edge in G, add an edge in G'.
- If edge (u, v) exist in G, set the weight to zero, otherwise assign weight equal to one.

Building Example

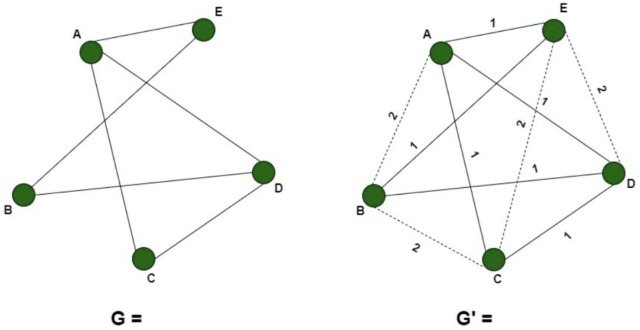

Figure: G and complimentary G'

Polynomial Reduction

The graph G has a Hamiltonian cycle if **there is** a cycle in G' passing through **all nodes only once with combined weight equal to zero**.

If the cycle passes through all nodes and the combined weight is zero, it means that the cycle **only contains edges present** in G. Hence, a **Hamiltonian cycle exists** in G.

If there is a Hamiltonian cycle in G, it also forms a **cycle** in G' with combined weight equal to zero. Hence, a **solution for TSP** exists in G'.

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3-SAT to Clique

Definition

A 3-SAT is composed from three-literal clauses. The goal is to reduce a clique of size k in a group of k clauses ϕ.

- Building a graph G of k clusters with a **maximum** of 3 nodes in each cluster;
- Each cluster corresponds to a **clause** in ϕ;
- Each node in a cluster is **labeled with a literal from the clause**;
- An edge is put between all pairs of nodes in different cluster **except for pairs of the form** $(x, \bar{x})$;
- **No edge is put between any pair of nodes in the same cluster**.

Building Example

Given the following clause:

$$\phi = (x_2 + x_1 + \bar{x}_3)(\bar{x}_1 + \bar{x}_2 + x_4)(x_2 + \bar{x}_4 + x_3)$$

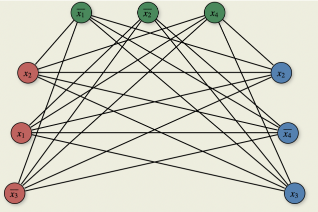

Figure: 3-SAT to clique

Building Example

If **two nodes are connected**, it means that the literal can be simultaneously *true*.

If **two literals**, not in the same clause can be assigned *true* simultaneously; hence, the nodes are also connected.

Polynomial Reduction

G has k-size clique, if ϕ is satisfiable.

If G has a clique of size k, the clique has **exactly one node** in from each cluster. Hence, all corresponding literals can be assigned *true* with each literal belong to an **individual** k **clauses**. Then, ϕ is **satisfiable**.

If ϕ is **satisfiable**, there is a combination of nodes corresponding to it. Let the set of nodes be A. From each clause, there are some literals that are *true*, that there are also in A. Remembering that **two literals cannot be from the same clause**, a clique can be formed by connecting a single node from each clause forming a **clique**.

Independent Set and Vertex Cover

Definition

Both problems can be traced to **covering** problems.

If a graph G has an **independent set** S, *it also has a* **vertex cover** $V - S$.

Polynomial Reduction

If S is an independent set, there is no edge $(u, v) \in G$, such that both v and u are in S. Therefore, either v or u **has to be in** $V - S$.

If $V - S$ is a vertex cover, between any pair of nodes $u, v \in S$, the edge connecting them **would not exist** in $V - S$, otherwise it violates the definition of such vertex cover. Hence, no pair in S can be reached by a single edge, creating an independent set.

Remark: Independent Set of size k corresponds to a Vertex Cover of size $V - |k|$.


thank you