MS-A0402
Foundations of discrete mathematics

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April 19, 2024

Teachers

- **Lecturer:** Jukka Kohonen
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Each exercise group is lead by one of the HA or TA.

Contacting the teachers:

- (Recommended) Zulip
  https://ms-a0402-2024.zulip.aalto.fi/
- (Secondary) E-mail, always include "MS-A0402" on the subject line

Weekly schedule (6 weeks)

- **Exercise sessions A**, Monday–Tuesday
  Homework based on previous week's lectures (+everything before)
- **Lectures**, Wednesday & Thursday 8-10, Hall D
- **Exercise sessions B**, Wednesday–Friday
  Homework based on the Wednesday lecture (+everything before)

Lecture notes, exercises, solutions (and everything) in
https://mycourses.aalto.fi/course/view.php?id=40608

Caveat: Group H03 has the B session already on Wednesday, so H03 recommended for those who can study the material in advance from the lecture notes.

Also: In **Laskutupa** you can solve homework and ask for help from various teachers: https://math.aalto.fi/en/studies/laskutupa/
(Mon–Fri almost 10–18, see exact schedule online)

Literature

- **Kenneth Rosen:** *Discrete Mathematics and its Applications*.
- (Kenneth Bogart: *Combinatorics Through Guided Discovery*.)
- (Richard Hammack: *Book of Proof*.)
- **Lecture notes** Available on the course page, updated during the course

Course content

- Set theory and formal logic
- Relations and equivalence
- Enumerative combinatorics
- Graph theory
- Modular arithmetics, elementary number theory

But more importantly:

- The fundamental notions, notations and methods of mathematics (definition, theorem, proof, example...)

Part 1: Sets and formal logic

Sets

- All mathematical structures are sets, and all statements about them can be described in terms of sets.

Example

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of integers.
- $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ is the set of rational numbers.
- $\mathbb{R}$ is the set of real numbers.
- $\{\Delta ABC : A, B, C \in \mathbb{R}^2\}$ is the set of triangles in the plane.
- The members (*elements*) of a set can be whatever:

$$A = \{\text{skateboard}, \text{paperclip}, 16, \pi, \text{infinity}\}$$

is a set.

Sets

- The most important notion in set theory is the symbol $\in$.
  - $x \in A$ if "the element x belongs to the set A".
  - $x \notin A$ if "the element x does not belong to the set A".

Example

- my car $\in \{\text{cars}\}$.
- $5 \in \mathbb{Z}$.
- $5 \in \mathbb{R}$.
- $5 \notin \mathbb{R}^2$.
- $\pi \in \mathbb{R}$.
- $\pi \notin \mathbb{Z}$.

Defining a set

- **Listing** the elements: $\{2, 4, 5, 7\}$, the set with elements 2, 4, 5, 7.
- **Ellipsis**: $\{10, 11, 12, \ldots, 20\}$, the set of integers from 10 to 20.
- **Set-builder notation**:

$$\{\texttt{expression} : \texttt{condition}\}$$

is a set containing all elements described by the `expression` such that the `condition` is satisfied for them.

  - $\{x^2 : x \in \mathbb{Z},\ 2 < x < 10\} = \{9, 16, 25, 36, 49, 64, 81\}$.
  - $\{x \in \mathbb{R} : -1 \leq x \leq 1\} = [-1, 1]$ (a closed interval of reals)

- **Special notation for empty set**: $\varnothing = \{\}$ is a set that has no elements.

Cardinality

- $|A|$ denotes the number of elements in a *finite* set A.
- This is called the **cardinality** of A.
- The cardinality is always a natural number (nonnegative integer).

Example

- $|\varnothing| = 0$
- $|\{\varnothing\}| = 1$
- $|\{a, b, c\}| = |\{a, c, c, b, a, c, b, b, a\}| = 3.$

But note that not all sets are finite (e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$). Later we will also define talk about cardinalities of infinite sets.

Equality of sets

- Two sets are the same if they contain the same elements.
  - For example: $\{2,3,4\} = \{4,2,4,3\}$.
  - Sets do not have "order", nor "multiplicity".
- Thus, there is only one "empty set" $\varnothing$.
- If $A = B$, then also $|A| = |B|$ (but not vice versa)

Proof techniques:

- To prove that $A = B$: Show that whenever $x \in A$, also $x \in B$. And show that whenever $x \in B$, also $x \in A$.
- To prove that $A \neq B$: One method just to exhibit one element that is on one of the sets but not in the other. Another method (for finite sets) is to show that the sets have different numbers of elements.

Subset

- $A \subseteq B$ ("A is a subset of B") if all elements of A are also in B.

  - e.g. $\varnothing \subseteq \{1, 2, 3\} \subseteq \mathbb{Z} \subseteq \mathbb{R}$.
  - $\varnothing$ is a subset of every set.
  - Every set is a subset of itself.
  - If $A \subseteq B$, then also $|A| \leq |B|$ (but not vice versa)
- So $A = B$ if

$$A \subseteq B \text{ and } B \subseteq A.$$

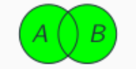
- If $A \subseteq B$ and $A \neq B$, then A is a *proper* subset of B.
  - Denoted $A \subsetneq B$, or sometimes $A \subset B$.

Proof techniques:

- To prove that $A \subseteq B$: Show that whenever $x \in A$, also $x \in B$.
- To prove that $A \nsubseteq B$: One method just to exhibit one element that is in A but not in B. Or (for finite sets) just show that A has more elements than B.

Elementary set operations

- Union: $x \in A \cup B$ if $x \in A$ *or* $x \in B$.



- Intersection: $x \in A \cap B$ if $x \in A$ *and* $x \in B$.

- Set difference: $x \in A \setminus B$ if $x \in A$ but $x \notin B$.

- Complement: $x \in A^c = \Omega \setminus A$ if $x \notin A$
  (but x is in the "universe" Ω, which is understood from context).

Set operations: Cartesian product

- $A \times B$ is the set of **ordered pairs**

$$\{(a, b) \ : \ a \in A, \ b \in B\}.$$

- $\{a, b, c\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.
- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ ("the xy-plane")

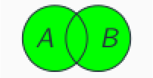
Set operations: Power set

- **Power set**: $P(A)$ is the set of all subsets of A.
- $P(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.
- $P(\{a,b,c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.
- $P(\varnothing) = \{\varnothing\} \neq \varnothing$.

Here we have sets whose *elements* are sets. Be careful that you understand what this means! For example, 1 is *not* an element of $\{\{1,2\},\{2,3\}\}$.

Cardinality of union

- If $|A| = 9$ and $|B| = 5$, what can we say about $|A \cup B|$?



  - $9 \leq |A \cup B|$.
  - $|A \cup B| \leq 14$.
  - $|A \cup B| \in \mathbb{N}$.
- In general, $|A \cup B| = |A| + |B| - |A \cap B|$.
- If $S \subseteq T$, then $|S| \leq |T|$.
- So

$$\max(|S|, |T|) \leq |S \cup T| \leq |S| + |T|.$$

Enumeration: Cardinality of Cartesian product

- Let $|S| = n$ and $|T| = m$.
- An ordered pair (s, t), where $s \in S$ and $t \in T$, can be chosen in nm ways.
- So $|S \times T| = nm = |S| \cdot |T|$.

Theorem

Let $A_1, \ldots, A_k$ be finite sets. Then

$$|A_1 \times \cdots \times A_k| = |A_1| \cdot \cdots \cdot |A_k|.$$

Enumeration: Cardinality of product set

- A subset A of $\{1, 2, \cdots, n\}$ is determined by, for each $1 \leq i \leq n$, whether or not $i \in A$.
- So a subset of $\{1, 2, \cdots, n\}$ can be described by a string of n *bits*: symbols 0 ("out") and 1 ("in").
- Example: The string 001101 corresponds to the set

$$\{3, 4, 6\} \subseteq \{1, \ldots, 6\}.$$

- We will talk more about *bits* and integers later on the course. The bit string 001101 can be understood as the integer $8 + 4 + 0 + 1 = 13$.

Enumeration: Cardinality of product set

- A subset of $\{1, 2, \cdots, n\}$ corresponds to a string of n symbols $0/1$, which is the same as an element of

$$\{0,1\}^n = \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n \text{ factors}}$$

- It follows that

$$|P(\{1,\ldots,n\})| = |\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

Theorem

Let A be a finite set. Then

$$|P(A)| = 2^{|A|}.$$

Useful properties

Subsets, unions and intersections have some properties that are almost "obvious", but very useful as "steps" in proofs.

Some examples: For any two sets A, B,

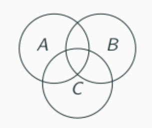
- $A \subseteq A \cup B$
- $A \cap B \subseteq A$
- $A \cap B = B \cap A$ (symmetry, or "commutativity")
- $A \cap B \subseteq A \cup B$

Make sure you understand why these are true (can you prove them from the elementary definitions?).

From the third one, it follows that the union never has fewer elements than the intersection. (Obvious?) Useful with so-called *Jaccard similarity*.

More useful properties

- Commutative laws:
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associative laws:
  - $(A \cap B) \cap C = A \cap (B \cap C)$
  - $(A \cup B) \cup C = A \cup (B \cup C)$
- Distributive law:
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- Proof via Venn diagrams (on blackboard).



Jaccard similarity and distance

How similar are two (finite) sets, if you look at their elements?

E.g. animals and plants described by *sets of features*. How similar are Cat and Dog? What about Seagull and Penguin?

$$
\begin{aligned}
\text{Cat} &= \{\text{tail}, \text{fourlegged}, \text{meows}, \text{breastfeeds}\} \\
\text{Dog} &= \{\text{tail}, \text{fourlegged}, \text{barks}, \text{breastfeeds}\} \\
\text{Seagull} &= \{\text{wings}, \text{layseggs}, \text{flies}\} \\
\text{Penguin} &= \{\text{wings}, \text{layseggs}\} \\
\text{Ostrich} &= \{\text{wings}, \text{layseggs}\} \\
\text{Platypus} &= \{\text{tail}, \text{fourlegged}, \text{layseggs}, \text{breastfeeds}\}
\end{aligned}
$$

Idea: count *how many common elements* they have (cardinality of intersection). Then normalize by how many they could share at most (cardinality of union).

Jaccard similarity and distance

Jaccard similarity J and distance d_J

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|}$$
$$d_J(A, B) = 1 - J(A, B)$$

(Work out the similarities of the animals.)

(Need a special definition when both sets empty. Then say similarity is 1, thus distance 0.)

Indexed family of sets

- Let $A_1, A_2, A_3, \cdots A_k \subseteq \Omega$ be sets.
- We say that
$$\{A_i : 1 \leq i \leq k\}$$
is an *indexed family of sets*
- $$\bigcup_{i=1}^{k} A_i = \{x \in \Omega : x \in A_i \text{ for some } 1 \leq i \leq k\}.$$
- $$\bigcap_{i=1}^{k} A_i = \{x \in \Omega : x \in A_i \text{ for every } 1 \leq i \leq k\}.$$
- This is union and intersection of more than two sets.

Indexed family of sets

Example

- Let $A_1 = \{0, 2, 5\}$, $A_2 = \{1, 2, 5\}$, $A_3 = \{2, 5, 7\}$.
- $$\bigcup_{k=1}^{3} A_k = \{0, 1, 2, 5, 7\}.$$
- $$\bigcap_{k=1}^{3} A_k = \{2, 5\}.$$

Indexed family of sets

- We can do the same for infinitely large families of sets.
- Let $A_1, A_2, A_3, \dots \subseteq \Omega$ be sets.
- We say that
$$\{A_i : i \geq 1\}$$
is an *indexed family of sets*
- $$\bigcup_{i=1}^{\infty} A_i = \{x \in \Omega : x \in A_i \text{ for some } i \in I\}.$$
- $$\bigcap_{i=1}^{\infty} A_i = \{x \in \Omega : x \in A_i \text{ for every } i \in I\}.$$

Indexed family of sets

Example

- Let $\Omega = \mathbb{R}$, and let A_k be the closed interval $A_k = [0, \frac{1}{k}]$ for $k \geq 1$.
- $$\bigcup_{k=1}^{\infty} A_k = [0, 1].$$
- $$\bigcap_{k=1}^{\infty} A_k = \{0\}.$$
- Proof on the blackboard.

Indexed family of sets

- We can do the same for other indexing sets as well. Let I be a set.
- Let $A_i \subseteq \Omega$ be a set, for each $i \in I$.
- $$\{A_i : i \in I\}$$

  is an *indexed family of sets*
- $$\bigcup_{i \in I} A_i = \{x \in \Omega : x \in A_i \text{ for some } 1 \leq i\}.$$
- $$\bigcap_{i \in I} A_i = \{x \in \Omega : x \in A_i \text{ for every } 1 \leq i\}.$$

Russel's paradox

- "A male barber in the village shaves the beards of precisely those men, who do not shave their own beard."

- Does the barber shave his own beard?
- Whether he does or does not, we get a contradiction.
- This is an instance of the problem of *self-reference* in set theory.

Russel's paradox

- For every man x in the village, there is a set S_x consisting of all the men whose beards he shaves.
- For the barber B,
$$S_B = \{x : x \notin S_x\}.$$
- In particular,
$$B \in S_B \Leftrightarrow B \notin S_B,$$
which is a contradiction!
- We are not allowed to use the set S in the formula that defines S!

Russel's paradox

- For every "universe" Ω and every statement P (without self-reference),
$$\{x \in \Omega : P(x)\} \subseteq \Omega$$
is a set.
- Let Ω be "the set of all sets", and let
$$S = \{A \in \Omega : A \notin A\}.$$
- Is S an element of itself? Again we get a contradiction.

Russel's paradox

To avoid this kind of contradictions, we decide:

- The "set of all sets" does not exist.
- No set is allowed to be an element of itself.
- All sets must be constructed from "safe and well-understood sets" (like $\mathbb{R}$) by taking
  - Subsets.
  - Cartesian products.
  - Power sets.
  - Unions.

Propositions (statements)

- A **proposition** (or *statement*) is a sentence that claims something, and is either true or false.
- We say that a proposition has a *truth value*, which is either "true" or "false". Commonly denoted by letters T, F or integers 1, 0.

  Let's use the integers: nice connection with arithmetic.
- Compare:
  - $5 + 3$ is an arithmetic expression, with integer value 8
  - $5 > 3$ is a logical expression (proposition), with truth value 1 (true)
- In mathematics, we are mostly interested in propositions that have a clear meaning and a well-defined truth value — whether or not we *know* the value, we think the value *exists* and is *in principle* knowable.

Some propositions

Example

- $2 \in \mathbb{Z}$
- $2 + 2 > 10$
- Sixty is divisible by three without remainder.
- The millionth decimal of π is 7.
- Every human (*Homo sapiens*) has two eyes.
- The housecat (*Felis domesticus*) is a mammal.
- Less than half of white clovers have four leaves.

Image: Vinayaraj
Wikimedia Commons

Observe: Propositions can be about math or about real world. Even purely mathematical claims might be expressed in words.

Some non-propositions

A "sentence" in natural language (e.g. English) is not necessarily a "proposition" in our sense.

Example

1. "Is $2+2=4$?" (question – does not claim a fact)
2. "Solve this equation!" (command – does not claim a fact)
3. "This sentence is false." (it is not possible for this sentence to have either truth value)
4. "x is an integer." (open sentence – we have not specified what x is)

Shades of definiteness

Real-world propositions often have some vagueness or ambiguity.

Example

- A million is a big number. (no clear boundary for "big")
- Orange juice tastes good. (opinion)
- Running is good for health. (in what sense? for whom, when?)
- There are many lakes in Finland. (perhaps, but what is "many"?)
- There are exactly 187 888 lakes in Finland. (what is a lake? when is a lake *in* Finland?)
- It rains right now in Espoo. (where? how many drops is rain?)

Usually we are fine with such claims, as long as we do not think they are more definite than they are. If necessarily, we can *make* them more definite (e.g. "by lake we mean this kind of waterbody").

Open and closed sentences

- Propositions are also called *closed sentences*.
- A **predicate** or *open sentence* is a sentence containing one or more variables (e.g. x, y), such that *if* we define their values, *then* the sentence becomes a definite proposition (true or false).
- For easy reference, we can give a *name* to a predicate, e.g. $P(x, y)$, where x, y are its variables (arguments).

Example

- $-1 \leq y \leq 1$.
- $5 \leq y \leq 2$.
- $E(x)$: x is an even integer.
- $P(x)$: the millionth decimal of π is x.
- $Q(n, x)$: the nth decimal of π is x.

"Closing" an open statement

There are two ways to convert an open sentence into a proposition. Let, for example, $P(x)$ be the open sentence "$x > 0$".

- **Assign** a value to the variables.
  - $P(5)$ is the proposition $5 > 0$ (whose value is true)
  - $P(-3)$ is the proposition $-3 > 0$ (whose value is false)
  - You can think of the open sentence P as a *function*, whose argument is (here) a number, and the value is a proposition, either true or false. Indeed they are sometimes called "propositional functions".
- **Quantify** over the variables.
  - "There exists a real number x such that $x > 0$" is a true proposition.
  - "For every real number x we have $x > 0$" is a false proposition.

Quantifiers

- "For all $x \in A$, $P(x)$ holds" is denoted formally

$$\forall x \in A : P(x).$$

- "There exists some $x \in A$, for which $P(x)$ holds" is denoted formally

$$\exists x \in A : P(x).$$

Note:
$\forall$, "for **A**ll", also called *universal quantifier*
$\exists$, "**E**xists", also called *existential quantifier*

Quantifiers

Example

- Which of the following propositions are true?
  - $\forall x \in \mathbb{R} : x^2 > 0$.
  - $\exists a \in \mathbb{R} : \forall x \in \mathbb{R} : ax = x$.
  - $\forall n \in \mathbb{Z} : \exists m \in \mathbb{Z} : m = n + 5$.
  - $\exists n \in \mathbb{Z} : \forall m \in \mathbb{Z} : m = n + 5$.
  - On every party, there are two guests who know the same number of other guests.
- 2 and 3 are true, 1 and 4 are false.
- We will revisit 5 later in the course.

Finite quantifying

A quantifier over a *finite* set can be understood as "and" or "or". Let, for example, $A = \{1, 2, 3, 4\}$, and $P(x)$ some predicate (eg. $x < 3$).

- $\forall x \in A : P(x)$ means that "$P(1)$ and $P(2)$ and $P(3)$ and $P(4)$" (we are claiming that all of these propositions are true)
- $\exists x \in A : P(x)$ means that "$P(1)$ or $P(2)$ or $P(3)$ or $P(4)$" (we are claiming that at least one of these propositions is true)

When quantifying over an infinite set (e.g. $\mathbb{N}$), this interpretation would require an infinitely long sentence, but at least mentally one can use this interpretation.

The $\exists$ quantifier says nothing about the number of suitable x's — just one is enough, but there could be more (perhaps even all).

Proving quantified sentences

"Easy" cases:

- $\exists x \in A : P(x)$ can be proven by exhibiting (just one) value of x that makes the claim true.
- $\forall x \in A : P(x)$ can be proven *false* by exhibiting (just one) value of x that makes the claim false!

"Difficult" cases:

- $\exists x \in A : P(x)$ can be proven false by proving that there isn't *any* x that would make $P(x)$ true.
- $\forall x \in A : P(x)$ can be proven true by proving that there isn't *any* x that would make $P(x)$ false.

If A is finite, one could tackle the difficult cases by simply *trying every possibility* and observing "I didn't find any such x". Otherwise we need some stronger tools $\rightarrow$ later on this course.

More than one quantifier

If a statement contains more than one quantifier, their order is crucial for the meaning! Consider following examples (all in integers).

Example

- $\exists x : \exists y : x + y = 7$ says we can choose an x, and then choose an y such that $x + y = 7$. By choosing $x = 3, y = 4$ we see this is true.
- $\forall x : \forall y : x + y = 7$ says it should be true no matter what x and y we choose. By choosing $x = 2, y = 3$ we see it is false.
- $\forall x : \exists y : x + y = 7$ is **true**; whatever x is chosen, we can *then* choose $y = 7 - x$, making the claim true. We have a "strategy" for the $\exists$, that works no matter what happens in the $\forall$.
- $\exists y : \forall x : x + y = 7$ is **false**: we cannot choose an x which would make $x + y = 7$ true for *all* y. (Elaborate!)

Observe: $\exists\exists$ can be swapped, $\forall\forall$ can be swapped, but $\exists\forall$ not.

Connectives

- Propositions can be composed with logical connectives:

negation	$\neg$	"not"
conjunction	$\wedge$	"and"
disjunction	$\vee$	"or"
implication	$\rightarrow$	"implies", "if ... then ..."
equivalence	$\leftrightarrow$	"if and only if"

"And" Connectives ($\wedge$, conjunction)

The meanings of connectives are *defined* via truth tables (cf. defining "times" by a multiplication table).

Let's start with the **and** connective $\wedge$, which connects two elementary propositions. For every possibility, we define the result.

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

- Observe: Connecting two propositions, so $2^2 = 4$ rows, one for each value combination.
- Think of a connective as an "operation" similar to arithmetic.
- In fact, $\wedge$ *is* a familiar arithmetical operation if our truth values are integers 0 and 1. Which one?

"Or" connective ($\vee$, disjunction)

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

$\vee$ claims that at least one of the elementary propositions is true (possibly both), so-called *inclusive or*.

E.g. "you can take this ride if you are at least 18 years or accompanied by someone who is" — we are not excluding adults who have company.

Think: what is the truth table for *exclusive or* ("exactly one of the elementary propositions is true")?

Think: can $\vee$ be seen as an arithmetic operation?

"Not" connective ($\neg$, negation)

A	$\neg A$
0	1
1	0

Negation simply reverses the truth value. Because it involves only *one* input, there are just two rows in the table.

Is it a simple arithmetic operation?

Equivalence, $\leftrightarrow$

Equivalence claims that the two elementary propositions have **the same truth value**, either both true or both false. Read "if and only if", or "is equivalent to".

A	B	$A \leftrightarrow B$
0	0	1
0	1	0
1	0	0
1	1	1

Often combined with quantifiers. For example:

$$\forall x \in \mathbb{N} : \Big((x^2 > 100) \leftrightarrow (x > 10) \Big)$$

Note. For some x both sides are true (e.g. $x = 11$), and for some x both are false (e.g. $x = 4$), but we cannot find an x where the sides have different truth. Thus the universal claim is true.

Implication, $\rightarrow$

- Implication $A \rightarrow B$ is a bit surprising.
- Can be understood as a "promise" that *if* A is true, then so is B.
- This promise is broken, or "false", if A is true but B is false.
- In all other three cases we say the promise holds (is true).

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

Implication, making sense of

Implication is often used *with a quantifier* (and then its meaning better matches the natural language "if").
Consider the claim "if x exceeds 3, then its square exceeds 9".

$$\forall x \in \mathbb{R} : \big((x > 3) \to (x^2 > 9)\big).$$

We have different kinds of cases:

- e.g. for $x = 4$, both sides are true
- e.g. for $x = 0$, both sides are false
- e.g. for $x = -4$, left side is false and right side is true

In fact all $x \in \mathbb{R}$ are similar. In all cases our "promise" holds.

Implication, making sense of

More examples:

"If your exam points are at least 50%, you pass the course."

- I'm not saying you couldn't pass with lower points.
- Equivalent contrapositive form: "If you don't pass the course, then your exam points are below 50%."

"If I am elected in March, the taxes will be lowered next year."

- I'm not saying what happens if I am not elected.
- Equivalent contrapositive form: "If taxes are not lowered next year, then I was not elected in March."

Tautologies

- A *tautology* is a (composed) proposition that is True regardless of the truth values of the elementary propositions that it is composed of.

Example

The following propositions are tautologies:

- $(\neg\neg P) \to P$ (double negation)
- $P \vee (\neg P)$ (excluded middle)
- $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ (contrapositive)
- $(P \leftrightarrow Q) \leftrightarrow ((P \to Q) \wedge (Q \to P))$ (equivalence law)
- These can be *proven* via truth tables (like on the blackboard).

- If $A \to B$ is a tautology (where A and B are composed propositions), then we write

$$A \Rightarrow B.$$

Tautologies

- This gives us a way to "calculate" with propositions.
- If $A \iff B$ (ie $A \leftrightarrow B$ is a tautology), then we can replace A by B everywhere in our logical reasoning.
- Often useful in math to replace an implication $P \to Q$ by its *contrapositive* $(\neg Q) \to (\neg P)$.

Example

The contrapositive (for $x \in \mathbb{R}$) of

$$\text{if } x^3 \neq 0 \text{ then } x \neq 0$$

is

$$\text{if } x = 0 \text{ then } x^3 = 0.$$

They claim the same thing. Do you find the latter easier to prove?

Treasures

Example

- Before you are three chests. They all have an inscription.
  - **Chest 1**: Here is no gold.
  - **Chest 2**: Here is no gold.
  - **Chest 3**: Chest 2 contains gold.

- We know that one of the inscriptions is true. The other two are false.
- If we can only open one chest, which one should we open?

Treasures

Example

- **Axiom:** One of the inscriptions is true. The other two are false.
- Let P_i be the proposition *"Chest i contains gold"*.
  - **Chest 1:** Here is no gold. $Q_1 := \neg P_1$
  - **Chest 2:** Here is no gold. $Q_2 := \neg P_2$
  - **Chest 3:** Chest 2 contains gold. $Q_3 := P_2$
- The axiom says

$$[Q_1 \wedge (\neg Q_2) \wedge (\neg Q_3)] \ \vee \ [(\neg Q_1) \wedge Q_2 \wedge (\neg Q_3)] \ \vee \ [(\neg Q_1) \wedge (\neg Q_2) \wedge Q_3]$$

Treasures

Example

- **Axiom:** One of the inscriptions is true. The other two are false.
- The axiom says

$$[Q_1 \wedge (\neg Q_2) \wedge (\neg Q_3)] \ \vee \ [(\neg Q_1) \wedge Q_2 \wedge (\neg Q_3)] \ \vee \ [(\neg Q_1) \wedge (\neg Q_2) \wedge Q_3]$$

$$\Longleftrightarrow$$

$$[(\neg P_1) \wedge (\neg\neg P_2) \wedge (\neg P_2)] \vee [(\neg\neg P_1) \wedge (\neg P_2) \wedge (\neg P_2)] \vee [(\neg\neg P_1) \wedge (\neg\neg P_2) \wedge P_2] \,.$$

$$\Longleftrightarrow$$

$$[\neg P_1 \wedge P_2 \wedge \neg P_2)] \vee [P_1 \wedge \neg P_2 \wedge \neg P_2] \vee [P_1 \wedge P_2 \wedge P_2] \,.$$

$$\Longleftrightarrow$$

$$[P_1 \wedge \neg P_2] \vee [P_1 \wedge P_2] \,.$$

$$\Longleftrightarrow$$

$$P_1$$

Treasures

- The axiom "One of the inscriptions is true. The other two are false." $\iff$ *"Chest 1 contains gold"*.
- **Lesson 1**: Open the first chest.
- **Lesson 2**: Manipulating propositions (by the tautology rule) is "mechanical". Mathematical reasoning *without quantifiers* can be automated.

Negation of quantifiers

- What is the negation (opposite) of

$$\forall x \in A : P(x)?$$

Example

- $A = \{\text{mathematicians}\}$, $P(x) =$ "x is bald".
- $\forall x \in A : P(x)$ means "all mathematicians are bald".
- The opposite is "some mathematicians are not bald".

So

$$\neg\forall x \in A : P(x)$$

is equivalent to

$$\exists x \in A : \neg P(x).$$

Computing with logical symbols

$$\begin{aligned} (\neg\neg P) &\iff P \\ (P \to Q) &\iff (\neg Q \to \neg P) \\ \exists x \in \Omega : \neg P(x) &\iff \neg \forall x \in \Omega : P(x) \end{aligned}$$

- In *constructive mathematics*, one only has the right implication

$$\exists x \in \Omega : \neg P(x) \Rightarrow \neg \forall x \in \Omega : P(x)$$

  in the last line.
- This is philosophically interesting, and also interesting in some algorithmic applications, but will not be relevant in this course.

Sets and predicate logic

- To any predicate $P(x)$ corresponds a set $\{x \in \Omega : P(x)\}$.
- To the set $S \subseteq \Omega$ corresponds the predicate $x \in S$.
- Sometimes mathematical statements are easier to think about in terms of sets, sometimes in terms of logical symbols.

Sets and predicate logic

- To any predicate $P(x)$ corresponds a set $S_P = \{x \in \Omega : P(x)\}$.
- To the predicate $P(x) \wedge Q(x)$ corresponds the set

$$\begin{aligned} S_{P\wedge Q} &= \{x \in \Omega : P(x) \text{ and } Q(x)\} \\ &= \{x \in \Omega : P(x)\} \cap \{x \in \Omega : Q(x)\} = S_P \cap S_Q. \end{aligned}$$

- To the predicate $P(x) \vee Q(x)$ corresponds the set

$$\begin{aligned} S_{P\vee Q} &= \{x \in \Omega : P(x) \text{ or } Q(x)\} \\ &= \{x \in \Omega : P(x)\} \cup \{x \in \Omega : Q(x)\} = S_P \cup S_Q. \end{aligned}$$

Why formal logic?

- We learn formal logic:
  - To define *precise* meanings of "and", "not", "or",...
  - To transform complicated statements to equivalent but easier statements, so that we can ...
  - ... assure ourselves and others that a thing is true;
  - ... *understand* why a thing is true.
  - Because it is the glue that holds mathematical statements together.
- We do **not** learn it in order to:
  - Write everything in symbols $\vee, \wedge, \forall, \exists, \cdots$
- Formal logic is in the background of all mathematics, not the forefront.
- If one wants to go "fully formal": consider *mathematical logic*, *axiomatic set theory*, and *proof checkers* (computer programs that require and check fully formal proofs)

Defining even

Next we'll talk about *even* and *odd* integers. Let's define what we *mean*.

Definition

An integer n is *even* if there is an integer k such that $n = 2k$.

- After this we can say "m and n are even integers", and understand (and exploit) the mathematical meaning. (Beware of symbol clash! What is the "k" here? BLACKBOARD)
- Often a *mixture* of natural (but precise) language, and symbols. Easier than symbols only.
- Often (in definitions) we say "if" but mean "if and only if". A manner of speech — avoid outside definitions!
- Often an unspoken quantifier: we said "an integer", meaning "for all integers"

Defining even, more formally

Let's try to write the definition more formally.

Definition

We define the predicate $\mathrm{Even}(x)$ as follows:

$$\forall n \in \mathbb{Z} : \big(\mathrm{Even}(x) \leftrightarrow \exists k \in \mathbb{Z} : n = 2k\big).$$

Defining odd

Definition

An integer n is *odd* if there is an integer k such that $n = 2k + 1$.

We have now defined *both* "even" and "odd" as *existential* claims. What does it now mean to be "not even" or "not odd"? It means "there is no such k that..."

We could prove that every integer is indeed either even or odd, but not both. (But we'll postpone this.) We could use any arithmetical laws that we already know. Perhaps using proof techniques such as induction (later this lecture).

Guessing vs. knowing vs. understanding

Suppose we are interested in how the parities of n and n^2 are related. We could run a small "experiment". Evens are red and odds are black.

n	0	1	2	3	4	5	...
n^2	0	1	4	9	16	25	...

It "seems" that when n is even, so is n^2, and when n is odd, so is n^2. We *know* this (by our calculations) for $0 \leq n \leq 5$.

We don't know (yet) for $n = 6$ or $n = 1279$. Also, perhaps we don't understand "why" this rule would always hold.

A proof might help, both for knowing for sure, and for understanding.

Proof techniques

- In the most abstract version, a mathematical theorem has an *axiom* (or conjunction of axioms) P, and a conclusion Q.
- A *proof* consists of a sequence of statements such that each row is either
  - An axiom or a definition.
  - Tautologically implied by the previous rows.

    if previous rows say $p_1, \ldots, p_k$, and $(p_1 \wedge \cdots \wedge p_k) \to q$ is a tautology, then the next row may say q.
  - Obtained from previous lines by "quantifier calculus":

$$\forall x : \neg P(x) \Leftrightarrow \neg \exists x : P(x)$$
$$\exists x : \neg P(x) \Leftrightarrow \neg \forall x : P(x)$$

  - A special case of a previous row.

    if one row says $\forall x P(x)$, then the next row may say $P(c)$.
  - An existential consequence of previous rows.

    if one row says $P(c)$, then the next row may say $\exists x : P(x)$.

Proof techniques

- In the most abstract version, a mathematical theorem has an *axiom* (or conjunction of axioms) P, and a conclusion Q.
- Most mathematical proofs uses one of the following tautologies:
  - $(P \land (P \to Q)) \Rightarrow Q$ (Direct proof)
  - $(P \land (\neg Q \to \neg P)) \Rightarrow Q$ (Contrapositive proof)
  - $(P \land ((P \land \neg Q) \to \mathit{False}) \Rightarrow Q$ (Proof by contradiction)
  - $((P_1 \lor P_2) \land (P_1 \to Q) \land (P_2 \to Q)) \Rightarrow Q$ (Proof by cases)
- ...and / or the following ways to prove existence:
  - $P(c) \Rightarrow \exists x : P(x)$ (Constructive proof)
  - $(\neg P(c) \to \exists x : P(x)) \Rightarrow \exists x : P(x)$ (Nonconstructive proof)
- Next, we will see examples of all these proof techniques.

Direct proof

Example

For all even integers n, also n^2 is even.

Proof.

- Let n be an *arbitrary* even integer.
- That means $n = 2k$ for some integer k.
- Then
$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$
.
- Since $2k^2$ is an integer, this means that n^2 is even. $\square$

Direct proof

Example

For all odd integers n, also n^2 is odd.

Proof.

- Let n be an *arbitrary* odd integer.
- That means $n = 2k + 1$ for some integer k.
- Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

- Since $2k^2 + 2k$ is an integer, this means that n^2 is odd. $\square$

Contrapositive proof

Example

For all integers n, if n^2 is odd, then n is also odd.

Proof.

- First attempt (direct proof):
- $n^2 = 2k + 1$ for some integer k.
- So $n = \pm\sqrt{2k+1}$, and n is an integer.
- No obvious way to write $n = 2\ell + 1$.

Contrapositive proof

Example

For all integers n, if n^2 is odd, then n is also odd.

Proof.

- New attempt (contrapositive proof):
- Need to prove that if n is **not** odd, then n^2 is **not** odd.
- So assume $n = 2k$ even.
- Then $n^2 = 4k^2 = 2(2k^2)$ is even, so not odd.
- Thus, if n were odd, then n^2 must also be odd. $\square$

Proof by contradiction

Example

$\sqrt{2} \notin \mathbb{Q}$.

Proof.

- Assume the claim was not true, so $\sqrt{2} \in \mathbb{Q}$.
- Then we could write $\sqrt{2} = \frac{p}{q}$, where p and q are integers with no common divisor.
- Then $2q^2 = p^2$, so p^2 is even.
- So p is even, and we can write $p = 2r$, $r \in \mathbb{Z}$
- So $q^2 = \frac{p^2}{2} = 2r^2$ is even.
- Now p and q are both even. But this contradicts our assumption that they had no common divisor.
- Thus the assumption was false, so $\sqrt{2} \notin \mathbb{Q}$. $\square$

Proof by cases

Example

For all real numbers x, y, it holds that $|xy| = |x| \cdot |y|$.

- Recall:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Proof by cases

Example

For all real numbers x, y, it holds that $|xy| = |x| \cdot |y|$.

Proof.

- Three cases:
  - Both numbers ≥ 0, so $xy \geq 0$: $|xy| = xy = |x| \cdot |y|$.
  - Both numbers < 0, so $xy > 0$: $|xy| = xy = (-x)(-y) = |x| \cdot |y|$.
  - The numbers have different sign, so $xy \leq 0$. Without loss of generality (WLOG) $x < 0 \leq y$:

$$|xy| = -xy = (-x)y = |x| \cdot |y|.$$

- These cases cover all possibilities, so the claim is true for all $x, y \in \mathbb{R}$. □

Constructive existence proof

Example

There exist integers that can be written as a sum of two cubes in more than one way.

Proof.

- $$12^3 + 1^3 = 1728 + 1 = 1729 = 1000 + 729 = 10^3 + 9^3 \qquad \square$$

Nonconstructive existence proof

Example

There exist irrational numbers $x, y \notin \mathbb{Q}$ such that $x^y \in \mathbb{Q}$.

Proof.

- The number $a = \sqrt{2}^{\sqrt{2}}$ is of the form x^y, where $x = y = \sqrt{2} \notin \mathbb{Q}$.
- If a is not rational, then $a^{\sqrt{2}}$ is also of the form x^y, where $x = a \notin \mathbb{Q}$ and $y = \sqrt{2} \notin \mathbb{Q}$.
- But

$$a^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2 \in \mathbb{Q}.$$

- So either $x = y = \sqrt{2}$ is an example of numbers with the desired property, or $x = a$, $y = \sqrt{2}$ is.
- So some irrational numbers with this desired property exist. □

Induction proofs

- A proof technique that is very useful for number sequences (but also in many other parts of mathematics)
- **Goal:** Prove a statement $P(n)$ for all natural numbers $n \in \mathbb{N}$.
- **Technique:**
  - First (base case) prove the first case $P(0)$.
  - Then (induction step) prove that, for an arbitrary $m \in \mathbb{N}$,
    IF $P(m)$ holds, THEN $P(m+1)$ also holds.
  - These two steps together prove that the statement $P(n)$ holds for any $n \in \mathbb{N}$.

$$P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \cdots .$$

Induction proofs

Example

Let a_n be recursively defined by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$. Then $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Proof.

- Base case: $a_0 = 0 = 1 - 1 = 2^0 - 1$, so the statement is true for $n = 0$.
- Induction step: Assume (*induction hypothesis*) that $a_m = 2^m - 1$. Then

$$a_{m+1} \overset{def}{=} 2a_m + 1 \overset{IH}{=} 2 \cdot (2^m - 1) + 1 = 2^{m+1} - 2 + 1 = 2^{m+1} - 1,$$

  so the statement is also true for $n = m + 1$.
- It follows that the statement $a_n = 2^n - 1$ is true for all $n \in \mathbb{N}$. $\square$

Induction proofs

Example

Prove that, for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Proof.

- Base case ($n = 0$):

$$\sum_{i=1}^{0}(2i-1) = \sum_{i\in\varnothing}(2i-1) = 0 = 0^2.$$

Induction proofs

Example

Prove that, for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Continued.

- Induction step: Assume (IH) that $\sum_{i=1}^{m}(2i-1) = m^2$. Then

$$\begin{aligned}\sum_{i=1}^{m+1}(2i-1) &\stackrel{def}{=} (2(m+1)-1) + \sum_{i=1}^{m}(2i-1) \\ &\stackrel{IH}{=} m^2 + 2(m+1) - 1 = m^2 + 2m + 1 = (m+1)^2,\end{aligned}$$

so the statement is also true for $n = m+1$. □

Induction proofs

- **Goal:** Prove a statement $P(n)$ for all natural numbers $n \in \mathbb{N}$.
- **More general technique:**
  - First (base case) prove the k first cases $P(0), \dots, P(k)$.
  - Then (induction step) prove that, for an arbitrary $m \in \mathbb{N}$,
    IF $P(m-k), \dots, P(m)$ holds, THEN $P(m+1)$ also holds.
  - These two steps together prove that the statement $P(n)$ holds for any $n \in \mathbb{N}$.

    $$(P(0) \wedge \cdots \wedge P(k)) \Rightarrow (P(1) \wedge \cdots \wedge P(k+1)) \Rightarrow (P(2) \wedge \cdots \wedge P(k+2)) \Rightarrow \cdots .$$

  - How large k needs to be, may depend on the problem.

Induction proofs

Example

The Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$. For all $n \in \mathbb{N}$ holds $f_n < 2^n$.

Proof.

- Base case: $f_0 = 0 < 1 = 2^0$ and $f_1 = 1 < 2 = 2^1$.
- Induction step: Assume (*induction hypothesis*) that $f_m < 2^m$ and $f_{m-1} < 2^{m-1}$. Then

$$f_{m+1} \stackrel{def}{=} f_m + f_{m-1} \stackrel{IH}{<} 2^m + 2^{m-1} < 2 \cdot 2^m = 2^{m+1},$$

  so the statement is also true for $n = m + 1$.
- It follows that the statement $f_n < 2^n$ is true for all $n \in \mathbb{N}$. $\square$

More about knowing and understanding

A famous problem from 1852 is the Four Color Problem.

Question

Is it true that any division of the plane into connected regions ("countries") can be colored in four colors, so that two regions sharing a boundary do not use the same color?

- Many failed attempts to prove (positive or negative).
- Famous *flawed* proof (positive) by Kempe in 1879. Flaw noticed 11 years later by Heawood.
- First complete (?) proof (positive) by Appel and Haken in 1976. Using "proof by cases" — in fact 1 834 cases, with a computer. Much discussion about "is it a valid proof? is it a good proof"?
- Later more formal computer-assisted proofs (Werner and Gonthier 2005, using Coq)
- Still no "simple" proof known. Perhaps we can claim we "know" it is true, but how well do we understand it?

Relations

- Relations are used in all parts of mathematics.
- Important applications outside of mathematics: Relational databases, automated translation,...

Example

- $y = x^2$. $\quad x, y \in \mathbb{R}$.
- $S \subseteq T$. $\quad S, T \in P(\Omega)$.
- $5|x - y$, *i.e.* $x \equiv y \pmod 5$. $\quad x, y \in \mathbb{Z}$.
- x and y are siblings. $\quad x, y \in \{\text{humans}\}$.
- $x \leq y$. $\quad x, y \in \mathbb{R}$.
- $x|y$, *i.e.* y is divisible by x. $\quad x, y \in \mathbb{Z}$.

Relations

- A *relation* can be defined in any of two different ways (which we will use interchangably):
  - A relation on a set A is a subset $R \subseteq A \times A$.
  - A relation is an open statement $R(x, y)$ that has a truth value for every $x, y \in A$.
- Recall: To the *predicate* $R(x, y)$ corresponds the *set*

$$\{(x, y) \in A^2 : R(x, y)\}.$$

This set is sometimes also denoted R.

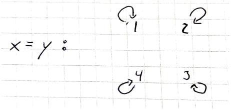
Relations

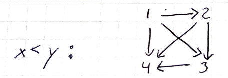
Example

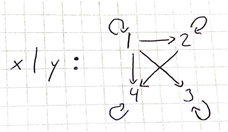
- Let $A = \{1, 2, 3, 4\}$.
- The equality relation $x = y$ on A is given by the set
$$\{(1,1),(2,2),(3,3),(4,4)\} \subseteq A^2.$$
- The order relation $x < y$ on A is given by the set
$$\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\} \subseteq A^2.$$
- The divisibility relation $x|y$ on A is given by the set
$$\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\} \subseteq A^2.$$

Relations

- A relation R on A can also be represented by a *directed graph*.
  - *Nodes* corresponding to the elements $x \in A$.
  - *Arcs* $x \to y$ if $R(x, y)$ holds.







Relations

- A relation on a set A is a subset $R \subseteq A^2 = A \times A$.
- Question: If

$$|A| = n,$$

  how many relations are there on A?
- Answer: $|P(A^2)| = 2^{|A \times A|} = 2^{|A| \cdot |A|} = 2^{n^2}$ different relations.

Relations

- We can also define a relation "from a set A to a set B":
  - As a subset $R \subseteq A \times B$.
  - As an open statement $R(x, y)$ that has a truth value for every $x \in A, y \in B$.

Example

- $x \in S$. $\quad x \in \Omega,\ S \in P(\Omega)$.
- x has shoes in size y. $\quad x \in \{\text{humans}\},\ y \in \mathbb{R}$.
- x is born in year n. $\quad x \in \{\text{humans}\},\ n \in \mathbb{N}$.

Relations

Definition

A definition $\sim$ on A is called:

- *reflexive* if
$$\forall x \in A : x \sim x.$$
- *symmetric* if
$$\forall x, y \in A : x \sim y \leftrightarrow y \sim x.$$
- *antisymmeric* if
$$\forall x, y \in A : (x \sim y \wedge y \sim x) \to x = y.$$
- *transitive* if
$$\forall x, y, z \in A : (x \sim y \wedge y \sim z) \to x \sim z.$$

Relations

Definition

A relation $\sim$ on A is called:

- *reflexive* if

$$\forall x \in A : x \sim x.$$

Example

- $x \leq y$ on $\mathbb{R}$
- $x|y$ on $\mathbb{Z}$
- $x = y$ on any set
- $x \equiv y \pmod{n}$ on $\mathbb{Z}$
- NOT reflexive: $x < y$ on $\mathbb{R}$
- NOT reflexive: x is a father of y on $\{\text{humans}\}$

Relations

Definition

A relation $\sim$ on A is called:

- *symmetric* if

$$\forall x, y \in A : x \sim y \leftrightarrow y \sim x.$$

Example

- x and y are siblings — on $\{\text{humans}\}$
- $|x - y| \leq 1$ — on $\mathbb{R}$
- NOT symmetric: $x - y \leq 1$ — on $\mathbb{R}$

Relations

Definition

A relation $\sim$ on A is called:

- *antisymmeric* if

$$\forall x, y \in A : (x \sim y \wedge y \sim x) \to x = y.$$

Example

- $x \leq y$ $\qquad x, y \in \mathbb{R}$
- $S \subseteq T$ $\qquad S, T \in P(\Omega)$

Relations

Definition

A relation $\sim$ on A is called:

- *transitive* if

$$\forall x, y, z \in A : (x \sim y \wedge y \sim z) \rightarrow x \sim z.$$

Example

- $x - y \in \mathbb{Z}$ $\qquad x, y \in \mathbb{R}$
- $x \leq y$ $\qquad x, y \in \mathbb{R}$
- NOT transitive: x and y have a parent in common. $\qquad x, y \in \{\text{Humans}\}.$

Equivalence relations

Definition

A relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example

- $x = y$ on any set.
- $x \equiv y \pmod{n}$ $x, y \in \mathbb{Z}$.
- $x - y \in \mathbb{Z}$ $x, y \in \mathbb{R}$.
- $|S| = |T|$ $S, T \in P(\Omega)$.
- x and y have the same biological mother $x, y \in \{\text{Humans}\}$.
- NOT an equivalence relation: $x \leq y$ $x, y \in \mathbb{R}$.
- NOT an equivalence relation: $|x - y| \leq 1$. $x, y \in \mathbb{R}$.

Equivalence relations

- An equivalence relation usually describes "sameness" in some sense.
- Every equivalence relation on A divides A into disjoint *equivalence classes* of elements that are "same".

Definition

- Let $\sim$ be an equivalence relation on A.
- The equivalence class of $a \in A$ is

$$[a] = [a]_\sim = \{x \in A : x \sim a\}.$$

Equivalence relations

Definition

- Let $\sim$ be an equivalence relation on A.
- The equivalence class of $a \in A$ is

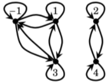
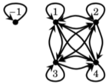
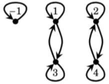
$$[a] = [a]_\sim = \{x \in A : x \sim a\}.$$

Example

- Let $\sim$ be congruence modulo 2, on $\mathbb{Z}$.
- $x \equiv y$ if $2|x - y$.
- Then

$$[0] = \{\dots, -4, -2, 0, 2, 4, \dots\} \text{ and } [1] = \{\dots, -3, -1, 1, 3, \dots\}.$$

Equivalence relations

Relation R	Diagram	Equivalence classes (see next page)
"is equal to" (=) $R_1 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4)\}$		$\{-1\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$
"has same parity as" $R_2 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(-1,1),(1,-1),(-1,3),(3,-1),$ $(1,3),(3,1),(2,4),(4,2)\}$		$\{-1,1,3\}$, $\{2,4\}$
"has same sign as" $R_3 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(3,4),$ $(4,3),(2,3),(3,2),(2,4),(4,2),(1,3),(3,1)\}$		$\{-1\}$, $\{1,2,3,4\}$
"has same parity and sign as" $R_4 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(1,3),(3,1),(2,4),(4,2)\}$		$\{-1\}$, $\{1,3\}$, $\{2,4\}$

Equivalence relations

Theorem

- *Let $\sim$ be an equivalence relation on A, and let $x, y \in A$.*
- *If $x \sim y$, then $[x] = [y]$.*
- *If $x \not\sim y$, then $[x] \cap [y] = \varnothing$.*

Proof.

- Blackboard $\square$

Equivalence relations

Theorem

- *Let $\sim$ be an equivalence relation on A, and let $x, y \in A$.*
- *If $x \sim y$, then $[x] = [y]$.*
- *If $x \not\sim y$, then $[x] \cap [y] = \varnothing$.*

- This shows that the equivalence classes form a *partition* of A: Every element in A is in exactly one equivalence class.

Definition

A partition of a set A is a collection of subsets $A_i \subseteq A$, $i \in I$ such that:

- $A = \bigcup_{i \in I} A_i$.
- $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Equivalence relations

- How many equivalence relations are there on a set with n elements.
- This is the Bell number B_n. (outside the scope of this course)
- The first few Bell numbers are

$$B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203, B_7 = 877.$$

- The numbers can be computed recursively in a *Bell triangle*.
- No “closed formula” known.

Partial orders

Definition

A relation $\preceq$ on A is an *order relation* if it is reflexive, antisymmetric, and transitive.

Example

- $x \leq y$ on $\mathbb{R}$
- $x|y$ on $\mathbb{N}$
- $S \subseteq T$ on $P(\Omega)$.

- An order relation is sometimes called a *partial order*.
- If $a \preceq b$ and $a \neq b$, then we write $a \prec b$.

Partial orders

Definition

- Let $\preceq$ be an order relation on A.
- Let $a, b \in A$ be elements such that:
    - $a \prec b$
    - $\neg \exists x \in A : a \prec x \prec b$.
- Then we say that b *covers* a, written $a \lessdot b$.

Example

- $18 \lessdot 19$ in the order $(\mathbb{Z}, \leq)$.
- $3 \lessdot 6$ in the order $(\mathbb{Z}, |)$.
- $\{a, b, c\} \lessdot \{a, b, c, d\}$ in the order $(P(\Omega), \subseteq)$.
- In the order $(\mathbb{R}, \leq)$, there are no covering pairs $a \lessdot b$.

Partial orders

Theorem

- *Let $\preceq$ be an order relation on a* finite *set A, $a, b \in A$.*
- *$a \prec b$ if and only if there exist $a_1, a_2, \ldots, a_n \in A$ such that*

$$a \lessdot a_1 \lessdot a_2 \lessdot \cdots \lessdot a_n \lessdot b.$$

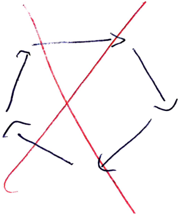
Proof.

Blackboard. □

- In other words, the order relation is uniquely defined if we know the corresponding covering relation
- Note: This is not true if A is infinite.

Hasse diagram

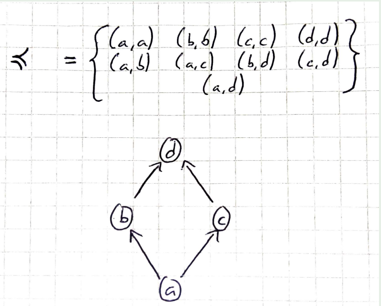
- So we can represent a finite order relation $(A, \preceq)$ as a directed graph where we only draw the arcs corresponding to covering pairs:
  - Nodes are elements of A.
  - Arc $a \to b$ if $a \lessdot b$.
- Because of antisymmetry, this graph has no *directed cycles*:



Hasse diagram

- When there are no directed cycles, we can draw the directed graph so that all arcs point upwards
- This representation of a finite order relation is called its *Hasse diagram*.

Example

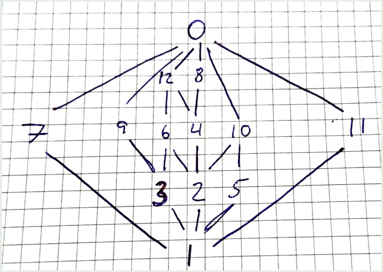


Hasse diagram

- The head of the arcs are usually not drawn in the Hasse diagram, as we already know that the arcs point upwards.

Example

The divisibility relation on $\{0, 1, 2, \ldots, 12\}$.



Linear extensions

- An order relation is called *linear*, or *total*, if for every x, y holds that $x \leq y$ or $y \leq x$.
- A totally ordered set is also called a *chain*.

Example

- The ordinary order relation $(\mathbb{N}, \leq)$ is linear, because for every two integers, if they are not the same, then one is smaller than the other.
- The divisibility relation $(\mathbb{N}, |)$ is not linear, because (for example) $5 \not| 7$ and $7 \not| 5$.

Linear extensions

- A linear relation $\leq$ on a set P is *compatible* with a partial order $\preceq$ on the same set, if for every $x, y \in P$ such that $x \preceq y$, also holds that $x \leq y$.
- We say that $\leq$ is a *linear extension* of $\preceq$

Example

- The ordinary order relation on $\{1, 2, 3, 4\}$ is a linear extension of the partial order

$$1 \preceq 2, 1 \preceq 3, 1 \preceq 4, 2 \preceq 4, 3 \preceq 4.$$

- Another linear extension of the same partially ordered set would be

$$1 \leq 3 \leq 2 \leq 4.$$

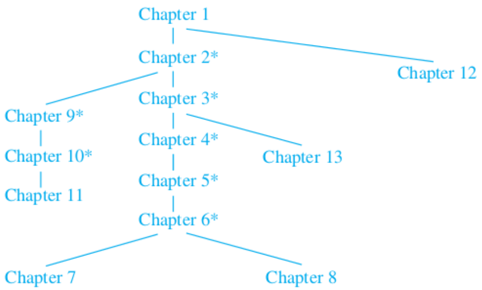
Linear extensions

Example

- The ordinary order relation on $\mathbb{N} \setminus \{0\} = \{1, 2, 3, 4, \dots\}$ is a linear extension of the divisibility relation.
  - A positive integer can never be divisible by any larger integer
- The ordinary order relation on $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is **not** a linear extension of the divisibility relation.
- Zero is divisible by any positive integer n (because $0 = 0 \cdot n$), although $0 \leq n$.

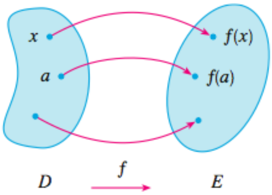
Linear extensions

- A partial order $\preceq$ can describe the dependencies of tasks. (Task T $\preceq$ Task S if the outcome of S is needed in order to begin T.)
- Then, a linear extension of $\preceq$ is an order in which the tasks can be performed.



Functions

- A function $f : A \to B$ is a relation "$f(x) = y$", such that for each element $a \in A$, there is a *unique* element $b \in B$ for which $f(a) = b$ holds.



- A is the *domain* of the function, and B is the *codomain*.
- The *range* of f is the set $f(A) \stackrel{\text{def}}{=} \{f(x) : x \in A\} \subseteq B$.

Functions

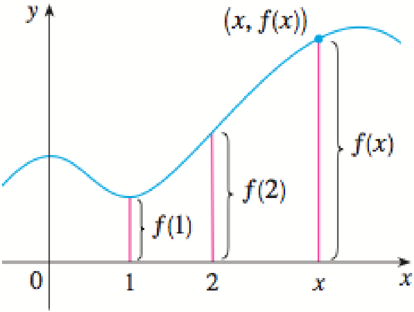
- Functions can thus be seen as a special case of relations:
    - Every element in the domain is related with some element in the codomain.
- A function f from A to B is compactly denoted $f : A \to B$.
- Sometimes a function does not need a name; in such case we write $a \mapsto b$ ("a maps to b") rather than $f(a) = b$.

Functions

- When considering a relation as a subset of $D \times E$, the set corresponding to f is its *graph*

$$\{(x, f(x)) : x \in D\} \subseteq D \times E.$$

- A function is often represented geometrically by its graph, especially when the domain and codomain are both (subsets of) $\mathbb{R}$.



Functions

Example

The function

$$\begin{aligned} f :& \mathbb{Z} \to \mathbb{Z} \\ & x \mapsto 4x + 5 \end{aligned}$$

(also written $f(x) = 4x + 5$) has:

- Domain (*määrittelyjoukko*) $\mathbb{Z}$.
- Codomain (*maalijoukko*) $\mathbb{Z}$.
- Range (*arvojoukko*)

$$\{4x + 5 : x \in \mathbb{Z}\} = \{\dots, -7, -3, 1, 5, 9, \dots\}.$$

- Graph (*kuvaaja*)

$$\{(x, y) : y = 4x + 5\} \subseteq \mathbb{Z}^2.$$

Equality of functions

The mathematical view to a function, from A to B, is as a relation between A and B, that is, a **collection of value pairs**:

$$\{(x, f(x)) \ : \ x \in A\} \subseteq A \times B.$$

Two functions f and g (both from A to B) are considered **same** (equal, identical, $f = g$) if their **values** agree, $f(x) = g(x)$, for every $x \in A$.

Details in how (by what expression, method, algorithm) the functions were defined does not matter. ($\neq$ the view in computer programming)

Example

All of the following functions $\mathbb{N} \to \mathbb{N}$ are the same:

- $f(x) = 2x$
- $g(u) = 2u$
- $h(x) = ((4x + 3) - 3)/2$
- $k(x) = |2x|$

Non-functions

A relation R, or a collection of pairs $(x, y) \in A \times B$, can **fail** to be a function in two ways:

- For some $x \in A$, there exists *no* $y \in B$ such that $R(x, y)$
- For some $x \in A$, there exist *several* $y \in B$ such that $R(x, y)$

A variant of the first is when we try to define a function by some "rule" of mapping x to y, but for some x, we have $y \notin B$!

Example

$f(x) = x - 5$ is not a function from $\mathbb{N}$ to $\mathbb{N}$.

If we have verified that some rule really gives a function with the intended domain and codomain, we often say that the function is **well-defined**.

Composition of functions

Two functions $f : A \to B$ and $g : B \to C$ can be *composed* into a function $g \circ f : A \to C$, $(g \circ f)(x) = g(f(x))$.

Example

The function $h(x) = 2(x + 3)$ can be written as $g \circ f$, where $f(x) = x + 3$ and $g(y) = 2y$.

Obs. notation: in $g \circ f$, it is meant that f is applied *first* (to the argument), and *then* g is applied to the result of f.

Composition is not commutative

It is not generally true that $g \circ f$ would be the same function as $f \circ g$.

Example

Consider $g(y) = 2y$ and $f(x) = x + 3$ (both $\mathbb{R} \to \mathbb{R}$).

- $h(x) = (g \circ f)(x) = g(f(x)) = 2(x+3) = 2x + 6$
- $k(x) = (f \circ g)(x) = f(g(x)) = (2x) + 3 = 2x + 3$

Clearly $h \neq k$, because they disagree at some (in fact many) points.

A single example is enough: $h(1) = 8 \neq k(1) = 5$.

Functions of many arguments

A function that “takes” two or more arguments can be understood a function **from a Cartesian product**, so it takes in “one” argument that is actually a pair or a tuple.

Example

Define $f : (\mathbb{R} \times \mathbb{R}) \to \mathbb{R} : (x, y) \mapsto x - y$.

Then $f(7, 2)$ can be understood as taking the argument $(x, y) = (7, 2)$ and giving the value $x - y = 5$.

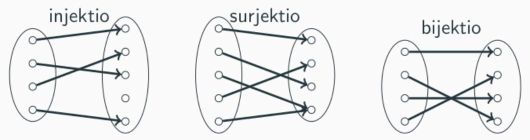
[There are other ways of defining such things, e.g. “currying”, where a function of one argument gives out a new function, to which the next argument is applied; often used in formal logic and computer science, but we won’t bother with that here.]

Injection, surjection, bijection

Definition

A function $f : A \to B$ is called

- *Injective* (or one-to-one) if
$$\forall x, y \in A : f(x) = f(y) \Rightarrow x = y.$$
- *Surjective* (or onto) if
$$\forall b \in B : \exists a \in A : f(a) = b.$$
- *Bijective* (or invertible) if it is injective and surjective.



Inverse functions

Definition

The *inverse* of the bijective function $f : A \to B$ is the function $g = f^{-1} : B \to A$ such that

$$f(a) = b \Longleftrightarrow g(b) = a.$$

- This defines the inverse function f^{-1} uniquely.
- If $f : A \to B$ is not bijective, then it can not have an inverse $B \to A$.
- Warning: Do not mistake the *function* f^{-1} for the *number* $f(x)^{-1} = \frac{1}{f(x)}$.
- The notation f^{-1} is also used in a *different* meaning (preimage of a set), which we shall discuss shortly.

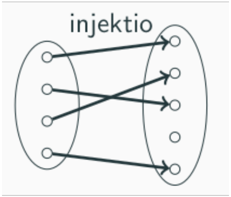
Why study functions and XXXjections?

- For math, obviously.
- For applications.
- Note: Relations & functions need not be numbers to numbers.
- E.g. assignment of jobs to workers, and ...
  - make sure every job gets done? ("function")
  - make sure no worker receives two jobs? ("injection")
  - make sure no worker is idle? ("surjection")
- Inverse functions often needed — bijection ensures it
- ALSO, a nice tool for comparing sets (NEXT TOPIC)

Cardinalities

Example

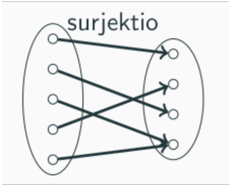
- Let A and B be finite sets.
- If there is an injection $A = \{a_1, \ldots, a_n\} \to B$, then $f(a_1), \ldots, f(a_n)$ are all *different* elements of B.
- So $A \to B$ injective $\Rightarrow n = |A| \leq |B|$.



Cardinalities

Example

- Let A and B be finite sets.
- If there is a surjection $A \to B = \{b_1, \ldots, b_m\}$, then there are *different* elements $a_1, \ldots a_m \in A$ such that $f(a_i) = b_i$ for $i = 1, \ldots, m$.
- So $A \to B$ surjective $|A| \geq |B| = m$.



Cardinalities

- For finite sets, there is an injective map $A \to B$ precisely if B has at least as many elements as A.
- For general sets, we take this as the *definition* of cardinality (i.e. "number of elements")

Definition

Let A and B be sets. We say that:

- $|A| = |B|$ if there exists a bijection $A \to B$.
- $|A| \leq |B|$ if there exists an injection $A \to B$.
- $|A| < |B|$ if $|A| \leq |B|$ **and not** $|A| = |B|$.

- Fact: There is a surjection $B \to A$ if and only if there is an injection $A \to B$.
- For finite sets this is relatively easy. For infinite sets, this requires a technical axiom about sets, called the axiom of choice. Do not worry about this.

Cardinalities

- $|A| = n$ if there is a bijection $A \to \{1, 2, \ldots, n\}$.
- The set A is *finite* if $|A| = n$ for some $n \in \mathbb{N}$. Otherwise it is *infinite*.
- For any infinite set A, there is an injection $\mathbb{N} \to A$. So $|\mathbb{N}| = \aleph_0$ is "the smallest infinite cardinality".
- The set A is *countable* if $|A| = |\mathbb{N}|$. If $|A| > |\mathbb{N}|$, then we say that A is *uncountable*.

Cardinalities

Theorem

- $|\mathbb{N}| = |\{0, 2, 4, 6, 8, \dots\}|$

Proof.

- Define $f : \mathbb{N} \to \{0, 2, 4, 6, 8, \dots\}$ by $f(n) = 2n$ for all $n \in \mathbb{N}$.
- Then f is a bijection.
- Inverse function $m \mapsto \frac{m}{2} \in \mathbb{N}$ for $m \in \{0, 2, 4, 6, 8, \dots\}$. $\square$

- Note: for infinite sets A, B, it is very possible that $|A| = |B|$ even when $A \subsetneq B$.

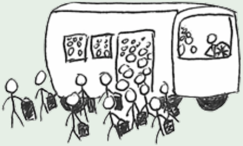
Infinite cardinalities

Example (Hilbert's hotel)

- David Hilbert is checking in to a hotel with infinitely many rooms (numbered $0, 1, 2, \ldots$)
- Unfortunately, every room is already occupied.
- Solution: All guests move rooms: The guest who used to stay in room k moves to room $k+1$ for all $i \in \mathbb{N}$.
- Now, Hilbert can move into room 0.

Infinite cardinalities

Example (Hilbert's hotel)



- The next day a bus arrives to the hotel, bringing infinitely (but countably) many new guests.
- Unfortunately, every room is already occupied.
- Solution: All guests move rooms: The guest who used to stay in room k moves to room $2k$ for all $i \in \mathbb{N}$.
- Now, the bus tourists can move into all odd numbered rooms.

Infinite cardinalities

Example (Hilbert's hotel)

- The next day, **infinitely** many buses (numbered $1, 2, 3, \dots$) arrive to the hotel, all bringing infinitely (but countably) many new guests.
- Solution: All previous guests move to odd numbered rooms.
- Now, the passengers on bus number k can move into rooms numbered $2^k, 2^k \cdot 3, 2^k \cdot 5, 2^k \cdot 7, \dots$.

Infinite cardinalities

Cardinalities

Theorem

The relation $|A| = |B|$ (between pairs of sets) is an equivalence relation (on $P(\Omega)$).

Proof.

- Reflexivity: The identity map $\iota : A \to A$ is a bijection.
- Symmetry: If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
- Transitivity: If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection. $\square$

Cardinalities

Theorem

- $|\mathbb{N}| = |\mathbb{Z}|$

Proof.

- Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(0) = 0, f(2k) = k \text{ and } f(2k-1) = -k \text{ for } k \geq 1.$$

- Then f is a bijection. $\square$

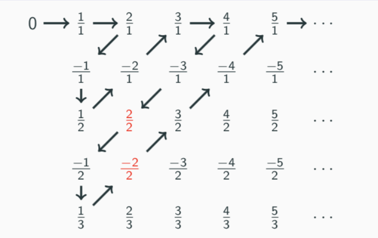
Cardinalities

Theorem

- $|\mathbb{N}| = |\mathbb{Q}|$

Proof.

- Order the numbers $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $q > 0$, as in the figure:



- Let $f(n)$ be the n^{th} "new" number in the sequence, for $n \in \mathbb{N}$.
- Then $f : \mathbb{N} \to \mathbb{Q}$ is a bijection. □

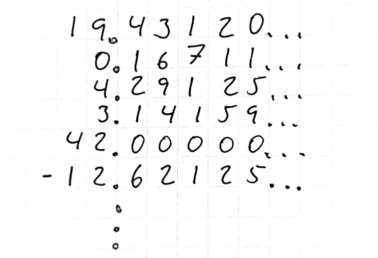
Cardinalities

Theorem

- $|\mathbb{N}| \neq |\mathbb{R}|$

Proof.

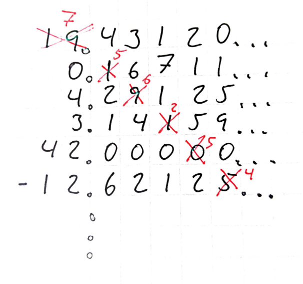
- Assume for a contradiction that we can "list" the real numbers as in the figure



Cardinalities

Continued.

- Change the i^{th} decimal digit of the i^{th} number, in any way you want.



- The "diagonal number" (in the example $7.56254\ldots$) was not in the original list.
- Contradiction, so $|\mathbb{N}| \neq |\mathbb{R}|$. $\square$

Cardinalities

- Recall: $|A| \leq |B|$ if there exists an injection $A \to B$.

Theorem

- $|A| \leq |B| \leq |C| \implies |A| \leq |C|$.

Proof.

- If $f : A \to B$ and $g : B \to C$ are injections, then $g \circ f : A \to C$ is an injection. $\square$

Cardinalities

Theorem (Not proved in this course)

- *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*
  - *This is a nice and challenging problem - Try it at home!*
- *For any sets A and B holds that $|A| \leq |B|$ or $|B| \leq |A|$.*
  - *This is a deep fact, and not true in* constructive mathematics - *Do not try it at home!*

Cardinalities — Summary

Sets can be roughly classified into **three classes** by cardinality.

- **Finite sets**. Inside this class, there are many different cardinalities.
- **Countably infinite sets**. This class contains surprisingly many sets, for example $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, "even integers", "squares of integers", "integers bigger than 100", $\mathbb{Z} \times \mathbb{Z}$, ..., **all of which have the same cardinality** (because $\exists$ bijections between them).
- **Uncountably infinite sets**. Some examples are $\mathbb{R}$, $\mathbb{R}^2$ and $P(\mathbb{N})$, which have (surprisingly) the same cardinality.
  OTOH there are uncountable sets with bigger cardinalities, for example $P(\mathbb{R})$ and $P(P(\mathbb{N}))$ (see exercises 3B6+3B8).

Part 2: Combinatorics

Principles of counting

Some basic principles are very useful in counting (finding finite cardinalities).

- **addition principle** (rule of sum): $A_1, \dots, A_k$ are pairwise disjoint, then
$$|A_1 \cup \cdots \cup A_k| = |A_1| + \cdots + |A_k|.$$
- **multiplication principle** (rule of product):
$$|A_1 \times \cdots \times A_k| = |A_1| \cdots |A_k|.$$
- **bijection**: If we can establish a bijection $A \to B$, then $|A| = |B|$.
- Recall that $|A| = m$ means (by definition) that there is a bijection $A \to \{1, 2, \dots, m\}$. In this light, the addition and multiplication principles are (easy, but not trivial) *theorems*.

Principles of counting

Example

- A bookshelf contains five physics books, seven chemistry books, and ten mathematics books. In how many ways can you choose two books about different subjects from the shelf?

Combining the rules

Example

- Let P, C, M be the sets of physics, chemistry, and math books respectively. $|P| = 5$, $|C| = 7$, $|M| = 10$.
- A pair of two books about different subjects is an element of

$$(P \times C) \cup (P \times M) \cup (C \times M).$$

- The number of choices is

$$\begin{aligned} & |(P \times C) \cup (P \times M) \cup (C \times M)| \\ = & |P||C| + |P||M| + |C||M| \\ = & 5 \cdot 7 + 5 \cdot 10 + 7 \cdot 10 \\ = & 155. \end{aligned}$$

Successive choices — Same possibilities each time

Using shorthand notation: $[n] = \{1, 2, \ldots, n\}$

Eg. counting tuples of integers (a, b, c), where each of a, b, c is an integer ranging from 1 to 10.

In other words, cardinality of $[10] \times [10] \times [10]$.

In other words, integer solutions (a, b, c) to the system of inequalities

$$0 \leq a \leq 10 \quad \wedge \quad 0 \leq b \leq 10 \quad \wedge \quad 0 \leq c \leq 10$$

By rule of product, the answer is $10 \cdot 10 \cdot 10 = 10^3$. More generally,

$$|A^k| = |A \times \cdots \times A| = (|A|)^k.$$

Successive choices — Decreasing possibilities

E.g. counting tuples of integers (a, b, c), each between 1 and 10, but *all different*.

- 10 possibilities for a.
- Whatever value a has, 9 possibilities left for b.
- Whatever values a, b have, 8 possibilities left for c.
- Rule of product: $10 \cdot 9 \cdot 8$ such tuples.

More generally, if initially we have n choices, and we choose k different items **in order**, the count is the **falling product**

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1).$$

Obs: exactly k factors in the product. The last is not $n-k$ (beware of fencepost error!)

Overcounting and adjustment

How many integer pairs (a, b) chosen from $[10]$, subject to extra requirement $a < b$?

Several methods, but one is **overcounting**.
(BLACKBOARD)

Counting linear orders

- In how many ways can we order the letters a,b,c in a linear order?
- abc, acb, bac, bca, cab, cba.
- The first letter could be chosen in 3 ways.
- Regardless of the first letter, the second letter can be chosen in 2 ways, and after this, the third letter can be chosen in only one way.
- So the number of linear orders is $3 \cdot 2 \cdot 1 = 6$

Counting linear orders

- In how many ways can we order n objects $a_1, a_2, \cdots, a_n$ in a linear order?
- The first object could be chosen in n ways.
- Regardless of the first i objects, the $(i+1)^{\mathrm{th}}$ object can be chosen in $(n-i)$ ways, $0 \leq i \leq n-1$.
- So the number of linear orders is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.
- This number is denoted $n!$, read "n **factorial**"
- By convention, $0! = 1$ ("the empty product"). Makes sense because there is **one** way to write a list of no objects: the empty list ( ).
- Also, we now have the **recurrence**

$$(n+1)! = n! \times (n+1)$$

valid for all $n \in \mathbb{N}$ (including zero, try it!). We could have *defined* factorials by starting from $0! = 1$ and the above recurrence defining all the rest.

Counting combinations

- In how many ways can we select a committee of 5 members from a party of 11?
- Call this number $\binom{11}{5}$. (read: "11 choose 5")
- If we also order the committee members, and order the non-members, we would get 11! possible orders total.
  - First committe member can be chosen in 11 ways, second committee member i 10 ways, ... , last committee member in 7 ways, first non-member in 6 ways, second non-member in 5 ways and so on.
- Every committee can be ordered in 5! ways, and the non-members can be ordered in 6! ways.
- We get $\binom{11}{5} \cdot 5! \cdot 6! = 11!$, so

$$\binom{11}{5} = \frac{11!}{6! \cdot 5!} = 462.$$

Counting combinations

- We can generalize this: How many "combinations" (subsets) of k elements are there in a set B of n elements?
- This number is denoted $\binom{n}{k}$. (read: "n choose k")
- The number of ways to select a set A with k elements and then order both A and $B \setminus A$ is

$$\binom{n}{k} \cdot k! \cdot (n-k)!,$$

but it is also $n!$ by the same argument as on the last slide.
- We get

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

Counting combinations

Example

- How many sequences of five cards (drawn from an ordinary 52 card deck) are there, if we know that it contains exactly two kings?
  - The word "sequence" impies that the order matters, so $\clubsuit 3, \heartsuit 5, \diamondsuit K, \clubsuit K, \heartsuit Q$ is a different sequence than $\heartsuit Q, \heartsuit 5, \diamondsuit K, \clubsuit 3, \clubsuit K$

Counting combinations

Example

$$\clubsuit 3, \heartsuit 5, \diamondsuit K, \clubsuit K, \heartsuit Q$$

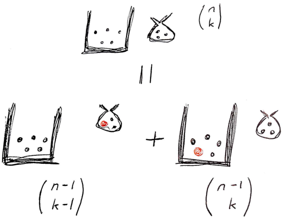
- The positions of the kings can be chosen in $\binom{5}{2}$ ways
- The first king can be chosen in 4 ways, the second king in 3 ways.
- The first non-king can be chosen in 48 ways, the next in 47 ways, and the last in 46 ways.
- By the multiplication principle there are

$$\binom{5}{2} \cdot 4 \cdot 3 \cdot 48 \cdot 47 \cdot 46 = 12453120$$

possible sequences.

Counting combinations

- There are $\binom{n}{k}$ ways to choose k balls from a box containing n balls.



- Refining according to whether or not our favourite (red) ball is chosen:

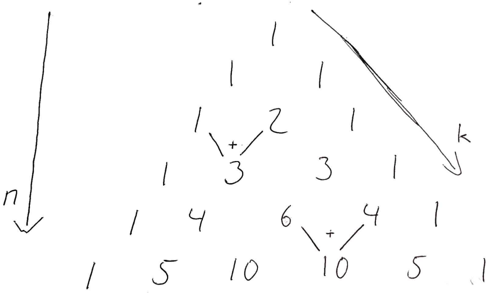
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Counting combinations

- We can also prove the same identity "algebraically":
- $$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!}+\frac{(n-1)!}{(n-1-k)!k!} \\ &= \frac{(n-1)!}{(n-1-k)!(k-1)!}\cdot\left[\frac{1}{n-k}+\frac{1}{k}\right] \\ &= \frac{(n-1)!}{(n-1-k)!(k-1)!}\cdot\frac{n}{(n-k)k} \\ &= \frac{n!}{(n-k)!k!} \\ &= \binom{n}{k}.\end{aligned}$$

Counting combinations

- Clearly, $\binom{n}{0} = \binom{n}{n} = 1$.
- So the *binomial coefficients* $\binom{n}{k}$ are the entries in the recursively defined *Pascal's triangle:*



Counting combinations

- Recall that, if $|A| = n$, then $|P(A)| = 2^n$:
- Order $A = \{a_1, a_2, \dots, a_n\}$.
- $\{0,1\}^n = \{0,1\} \times \cdots \times \{0,1\}$ is the set of length n bitstrings.
- Define $f : P(A) \to \{0,1\}^n$ by $f(S) = (f_1, \dots, f_n)$, where

$$f_i = \begin{cases} 1 & \text{if } a_i \in S \\ 0 & \text{if } a_i \notin S \end{cases}$$

- f is a bijection, so

$$|P(A)| = |\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

Counting combinations

- On the other hand, if $|A| = n$, then $P(A) = P_0 \cup P_1 \cup \cdots \cup P_n$, where

$$P_k = \{S \subseteq A : |S| = k\}.$$

- $|P_k| = \binom{n}{k}$, so

$$2^n = |P(A)| = \sum_{k=0}^{n} |P_k| = \sum_{k=0}^{n} \binom{n}{k}.$$

Counting combinations with repetition

Example

- A box contains (many) blue, red and green balls.
- In how many ways can I select 5 balls from this box, if the order does not matter?
- So ${\color{blue}\bullet}{\color{green}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}$ is the same selection as ${\color{blue}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{green}\bullet}$.

Counting combinations with repetition

Example (Continued)

- Solution: Represent any selection by always lining up the balls blue first, then red, then green.

$$\color{blue}{\bullet\bullet\bullet}\color{red}{\bullet}\color{green}{\bullet} \qquad \color{blue}{\bullet\bullet}\color{green}{\bullet\bullet\bullet} \qquad \color{red}{\bullet\bullet\bullet\bullet\bullet}$$

- If we separate the different colors by bars, then we can reconstruct the colors from the position of the bars.
- The three selections above are now represented as

$$\bullet\bullet\bullet|\bullet|\bullet \qquad \bullet\bullet||\bullet\bullet\bullet \qquad |\bullet\bullet\bullet\bullet\bullet|$$

- A selection is given by placing bars in two out of 7 positions in a sequence, and placing balls in the other 5 positions.
- So there are $\binom{7}{2}$ different selections.

Counting combinations with repetition

- More generally, assume we have n different kinds of balls, and want to select k from these.
- Like in the previous example, this can be represented by a configuration of k balls and $n-1$ bars ordered in a sequence.
- So there are
$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$
different ways to select.
- Note: This is also the number of non-negative integer solutions to the equation
$$x_1 + \cdots + x_n = k,$$
where x_i represents the number of balls of the i^{th} kind.

Binomial theorem

Theorem (Binomial theorem)

For all $n \in \mathbb{N}$ and all $x, y \in \mathbb{R}$ holds

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Combinatorial proof.

- Expand the product $(x+y)^n$ into a sum of 2^n monomial terms.
- Each term corresponds to a way to select either x or y from each of the n parentheses.
- The monomial term $x^k y^{n-k}$ corresponds to selecting x from k of the parentheses, and y from $n-k$ of the parentheses.
- This can be done in $\binom{n}{k} = \binom{n}{n-k}$ ways. $\square$

Binomial theorem

Theorem (Binomial theorem)

For all $n \in \mathbb{N}$ and all $x, y \in \mathbb{R}$ holds

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Induction proof.

- Base case $n = 0$:

$$(x+y)^0 = 1 = \binom{0}{0} x^0 y^{0-0}.$$

- Base case $n = 1$:

$$(x+y)^1 = x + y = \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k}.$$

Binomial theorem

Induction proof.

- Induction step: Assume true for $n = M$.
- Then

$$\begin{aligned}
(x+y)^{M+1} &= (x+y)(x+y)^M \\
&\overset{\text{IH}}{=} (x+y)\sum_{k=0}^{M}\binom{M}{k}x^k y^{M-k} \\
&= \sum_{j=0}^{M}\binom{M}{j}x^{j+1}y^{M-j} + \sum_{k=0}^{M}\binom{M}{k}x^k y^{M-k+1} \\
&= \sum_{k=1}^{M+1}\binom{M}{k-1}x^k y^{M-(k-1)} + \sum_{k=0}^{M}\binom{M}{k}x^k y^{M-(k-1)}
\end{aligned}$$

Binomial theorem

Induction proof.

$$= x^{M+1} + \sum_{k=1}^{M} \left(\binom{M}{k-1} + \binom{M}{k} \right) x^k y^{M+1-k} + y^{M+1}$$

$$= x^{M+1} + \sum_{k=1}^{M} \binom{M+1}{k} x^k y^{M+1-k} + y^{M+1}$$

$$= \sum_{k=0}^{M+1} \binom{M+1}{k} x^k y^{M+1-k}.$$

- By the induction principle,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \text{ for all } n \in \mathbb{N}. \quad \square$$

Binomial theorem

Example

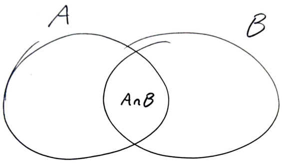
- This shows in a new way that

$$2^n = (1+1)^n = \sum_k \binom{n}{k} 1^k 1^{n-k} = \sum_k \binom{n}{k}.$$

- Similarly,

$$3^n = (2+1)^n = \sum_k \binom{n}{k} 2^k 1^{n-k} = \sum_k 2^k \binom{n}{k}.$$

Inclusion exclusion principle



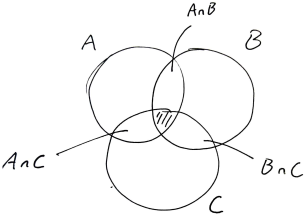
- The inclusion exclusion principle for two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example

- How many 8 bit strings start or end with two zeroes?
- Answer: $2^6 + 2^6 - 2^4 = 112$.

Inclusion exclusion principle for three sets



- The inclusion exclusion principle for three sets:

$$\begin{aligned}|A \cup B \cup C| = \quad & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C|.\end{aligned}$$

Inclusion exclusion principle for three sets

Example

- A martial arts club has courses in aikido, boxing and capoeira.
- There are 30 aikido students, 25 boxers and 35 capoeira dancers.
- 5 people do both aikido and boxing, 19 do both aikido and capoeira, and 7 boxers also do capoeira.
- One student (Chuck Norris) studies all martial arts at once.
- How many martial artists does the club have?

Inclusion exclusion principle for three sets

Example

- Let A, B and C be the sets of students of the respective martial arts.
- $|A| = 30$, $B = 25$, $|C| = 35$.
- $|A \cap B| = 5$, $|A \cap C| = 19$, $|B \cap C| = 7$
- $|A \cap B \cap C| = |\{\text{Chuck Norris}\}| = 1$
- The total number of martial artists is

$$\begin{aligned}
|A \cup B \cup C| = \quad & |A| + |B| + |C| \\
& - |A \cap B| - |A \cap C| - |B \cap C| \\
& + |A \cap B \cap C| \\
= \quad & 30 + 25 + 35 - 5 - 19 - 7 + 1 \\
= \quad & 60.
\end{aligned}$$

Inclusion exclusion principle for three sets

Example

- How many permutations a_1a_2, a_3, a_4 of the set $\{1, 2, 3, 4\}$ are such that $a_{i+1} \neq a_i + 1$ for all $i \in \{1, 2, 3\}$?
- In other words, the string a_1a_2, a_3, a_4 must not contain "12", "23", or "34".
- For example, the permutation 1432 satisfies the property, but the permutation 1423 does not.
- A permutation containing "12" can be thought of as a permutation of $\{`12', 3, 4\}$. There are $3! = 6$ such permutations.
- Similarly, there are $3! = 6$ permutations that contain "23", and $3! = 6$ permutations that contain "34".

Inclusion exclusion principle for three sets

Example

- Permutations that contain both "12" and "23" correspond to permutations of $\{`123', 4\}$. There are $2! = 2$, such permuations, namely 1234 and 4123.
- Similarily, there are 2 permutations that contain both "23" and "34", and 2 permutations that contain both "12" and "34".
- The only permutations that contains all the "forbidden pairs" is 1234.
- So there are

$$4! - 3 * 3! + 3 * 2! - 1 = 24 - 18 + 6 - 1 = 7$$

permutations with the desired property.

Inclusion exclusion principle

- In the three set case, denote
  - $s_1 = |A_1| + |A_2| + |A_3|$
    "count elements that are in one of the sets, one set at a time".
  - $s_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$
    "count elements that are in two sets, one pair of sets at a time".
  - $s_3 = |A_1 \cap A_2 \cap A_3|$
    "count elements that are in three sets, (one triple of sets at a time)".
- Then the inclusion exclusion principle says

$$|A_1 \cup A_2 \cup A_3| = s_1 - s_2 + s_3 = \sum_{k=1}^{3} (-1)^{k-1} s_k.$$

Inclusion exclusion principle

- For a collection of finite sets $A_1, \ldots, A_n$, let

$$s_k = \sum_{|B|=k} \left| \bigcap_{i \in B} A_i \right|,$$

where the sums are taken over subsets of $\{1, \ldots, n\}$.

Theorem

- *If $A_1, \ldots, A_n$ are finite sets, and $s_1, \ldots, s_k$ are as above, then*

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} s_k.$$

Inclusion exclusion principle

Theorem

- *If $A_1, \ldots, A_n$ are finite sets, and $s_1, \ldots, s_k$ are as above, then*

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} s_k.$$

Proof.

- Let $x \in A_1 \cup \cdots \cup A_n$, and let

$$I_x = \{i : x \in A_i\} \subseteq \{1, \ldots, n\}$$

be the indices of the sets containing x. Let $m = |I_x|$

- x belongs to the set $\bigcap_{i \in B} A_i$ if and only if $B \subseteq I_x$.

Inclusion exclusion principle

Proof.

- So on the right hand side, x is counted

$$\begin{aligned}\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1} &= -\sum_{k=1}^{m}\binom{m}{k}(-1)^{k} \\ &= 1-\sum_{k=0}^{m}\binom{m}{k}(-1)^{k-1} \\ &= 1-(1-1)^m = 1 \text{ times.}\end{aligned}$$

- Hence each element $x \in A_1 \cup \cdots \cup A_n$ is counted exactly once on each side of the equation

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n}(-1)^{k-1}s_k. \quad \square$$

Counting surjections

Let $n \geq m$, consider sets $[n] := \{1, \ldots, n\}$ and $[m] := \{1, \ldots, m\}$.

- How many ways to place n distinct balls in m boxes so that no box is empty?
- How many surjections exist $[n] \to [m]$?
- How many m-tuples from numbers $1, \ldots, n$, repetition allowed, and each number must appear? Eg. $n = 3$ and $m = 2$: tuples $112, 121, 122, 211, 212, 221$ (6 of them)

Same question in three forms! Let's denote this count $L(n, m)$.

Counting surjections

- For $i = 1, \ldots, m$, let A_i be the set of maps

$$\varphi : X \to \{1, \ldots, m\}$$

  that "miss i", i.e. $\varphi(x) \neq i$ for all $x \in X$.

- $A_{i_1} \cap \cdots \cap A_{i_k}$ is the set of maps

$$X \to \{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\}.$$

- $$|A_{i_1} \cap \cdots \cap A_{i_k}| = (m-k)^n.$$

- $$s_k = \sum_{|B|=k} \left| \bigcap_{i \in B} A_i \right| = \binom{m}{k}(m-k)^n.$$

Counting surjections

- The number of maps $X \to \{1, \ldots, m\}$ is m^n.
- The number of non-surjections is

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_m| &= \sum_{k=1}^{m} (-1)^{k-1} s_k \\
&= \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} (m-k)^n.
\end{aligned}
$$

- So the number of surjections is

$$
\begin{aligned}
L(n,m) &= m^n - \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} (m-k)^n \\
&= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (m-k)^n.
\end{aligned}
$$

Counting surjections

Example

- A secret Santa has brought 6 gifts to a christmas party with 4 guests.
- In how many ways can the gifts be distributed, so that all guests get at least one gift?
- This is the number of surjections from the set of gifts to to the set of guests.
- $$\begin{aligned} L(6,4) &= \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^6 \\ &= 4^6 - 4 \cdot 3^6 + 6 \cdot 2^6 - 4 \cdot 1^6 \\ &= 1560. \end{aligned}$$

Counting surjections

- The number of surjective maps $\{1,2,3,4,5,6\} \to \{1,2,3,4\}$ is

$$L(6,4) = 1560 = 24 \cdot 65.$$

- Is it a coincidence that $L(6,4)$ is divisible by $4! = 24$?

Surjections vs. partitions

- If we want to place 6 numbered balls into 4 numbered boxes, leaving no box empty, we can do it in two phases:
  1. Partition the balls into 4 nonempty parts. The parts do not have a "number", we simply note which balls belong together. Call the number of possible partitions $S(6, 4)$.
  2. Then number the 4 parts (so they become numbered boxes). This can be done in 4! ways.
  3. Apply the rule of product: $L(6, 4) = S(6, 4) \cdot 4!$.

$S(n, k)$ is called the *Stirling number of the second kind.*

Counting the partitions by recursion

$S(n,k)$: Given n numbered balls $1,\ldots,n$. How many ways to partition it into k nonempty parts?

- Base cases $S(n,1) = S(n,n) = 1$.
- Recursion for $1 < k < n$: Consider the last ball (number n). Two possibilities:
  - The last ball is its own part. Other $n-1$ balls can be partitioned into $k-1$ parts in $S(n-1,k-1)$ ways.
  - The last ball is with some other balls. First partition the other balls into k parts: $S(n-1,k)$ ways. Then put the last ball in one of these parts: k ways.
- Applying the rules of sum and product we get

$$S(n,m) = S(n-1,k-1) + k \cdot S(n-1,k)$$

Counting partitions by recursion

Stirling numbers $S(n,k)$ in a triangle:

	$k=1$	2	3	4	5	6	7	row sum
$n=1$	1							1
2	1	1						2
3	1	3	1					5
4	1	7	6	1				15
5	1	15	25	10	1			52
6	1	31	90	65	15	1		203
7	1	63	301	350	140	21	1	877

E.g.: $S(6,4) = S(5,3) + 4 \cdot S(5,4) = 25 + 4 \cdot 10 = 65.$

Note: Sum of nth row = total number of ways to partition a n-element set into *any* number of nonempty parts. This is the Bell number B_n we saw earlier.

Permutations

Definition

A bijection $\pi : A \to A$ from a set to itself is called a *permutation*.

Example

- Let $\pi : \{1,2,3,4\} \to \{1,2,3,4\}$ be defined by:

$$\pi_1 = 3, \pi_2 = 2, \pi_3 = 4, \pi_4 = 1.$$

- In *two line notation* this is denoted:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \cdots .$$

Permutations

- As a permutation is a bijection, it also has an inverse.
- In the two line notation, the inverse of a permutation is obtained by changing the place of the first and second row (and reordering the columns according to the first row).

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

$$\pi^{-1} = \begin{pmatrix} 3 & 2 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}.$$

Permutations

- Permutations can be composed as functions. Let

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

- The two line notation of the permutation $\sigma \circ \pi$ is computed as follows:

$$\sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

- The first two rows are aligned according to π; The last two rows according to σ.

Permutations

- $$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$
- $$\sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$
- $$\pi \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}.$$
- “Multiplication” $\pi\sigma = \pi \circ \sigma$ of permutations is not commutative ($\pi\sigma \neq \sigma\pi$).

Cycle notation

- Permutations can be represented by cycle notation.
- Consider

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 3 & 5 & 7 & 6 \end{pmatrix}.$$

- Here, $1 \mapsto 2 \mapsto 4 \mapsto 3 \mapsto 1$. This is a *cycle*, which is denoted (1243).
- Because $\alpha_5 = 5$, there is also a cycle (5).
- Finally, $6 \mapsto 7 \mapsto 6$, so there is a cycle (67) .
- On cycle notation we get

$$\alpha = (1243)(67) = (4312)(76) = (5)(1243)(67) = \cdots$$

Cycle notation

- The inverse of a cyclic permutation is easy to compute:

$$(a_1 \cdots a_k)^{-1} = (a_k \cdots a_1).$$

- In any group it holds that

$$(\pi \cdot \sigma)^{-1} = \sigma^{-1}\pi^{-1}.$$

- So for example, when

$$\pi = (145)(27)(3698),$$

we can compute

$$\pi^{-1} = (8963)(72)(541) = (154)(27)(3896).$$

All permutations of a set

- The set of **all** permutations of A is denoted S_A.
- The set of **all** permutations of $\{1, 2, \ldots n\}$ is denoted S_n.
- Note: $|S_A| = |A|!$, and $|S_n| = n!$.
- The **identity permutation**

$$\iota = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

  is such that $\iota\pi = \pi\iota = \pi$ holds for all $\pi \in S_n$.
- $$\pi^{-1}\pi = \pi\pi^{-1} = \iota.$$
- $$(\pi\sigma)\tau = \pi(\sigma\tau)$$

  holds for all $\pi, \sigma, \tau \in S_n$ *(associativity)*.

Cycle notation, example S_3

Example

- All permutations in S_3 can be represented by a single cycle (together with some trivial cycles):

$$
\begin{aligned}
123 &= (1)(2)(3) = \iota \\
132 &= (1)(23) = (23) \\
213 &= (12)(3) = (12) \\
231 &= (123) \\
312 &= (132) \\
321 &= (13)(2) = (13)
\end{aligned}
$$

Cycle notation, example S_4

- All permutations in S_n can be written as a product of *disjoint* cycles.
- If $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_\ell)$ are disjoint, then

$$(a_1, \ldots, a_k)(b_1, \ldots, b_\ell) = (b_1, \ldots, b_\ell)(a_1, \ldots, a_k)$$

Example

The permutations in S_4 are:

ι							
(12)	(13)	(14)	(23)	(24)	(34)		
(123)	(132)	(124)	(142)	(134)	(143)	(234)	(243)
(12)(34)	(13)(24)	(14)(23)					
(1234)	(1243)	(1324)	(1342)	(1423)	(1432)		

Permutation groups

- The set of permutations of $\{1, 2, \ldots n\}$ is denoted S_n.
- Note: $|S_n| = n!$.
- We often write $\pi \in S_n$ using *one line notation* (without parentheses):

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix} = \pi_1 \pi_2 \cdots \pi_n$$

All permutations, as a group

Definition (Group)

Let G be a set, and $\cdot : G \times G \to G$. The pair $(G, \cdot)$ is called a *group*, if the following holds:

- Associativity:
$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ for all } a, b, c \in G.$$
- Neutral element: There exists $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
- Inverse: For every $a \in G$, there exists $a^{-1} \in G$ such that
$$a \cdot a^{-1} = a^{-1} \cdot a = e.$$

The **symmetric group** $(S_n, \circ)$, whose elements are **all** $n!$ permutations of $[n]$, is a group, whose neutral element is the identity permutation ι.

Powers of a single permutation

We define all integer powers of π. If n is a positive integer,

$$\pi^n = \underbrace{\pi\pi\cdots\pi}_{n \text{ times}} \qquad \text{(composition)}$$

$$\pi^0 = \iota \qquad \text{(identity)}$$

$$\pi^{-n} = (\pi^n)^{-1} = (\pi^{-1})^n \qquad \text{(inversion)}$$

With these definitions, powers of a permutation π behave just like powers of numbers, e.g. $(\pi^a)(\pi^b) = \pi^{a+b}$, and $\pi^1 = \pi$.

It makes sense because $(\pi^a)(\pi^b)$ means "apply π first b times and then a more times".

Every element returns to itself, sooner or later

Let π be a permutation over a *finite* set A, with $|A| = n$.

What happens to a particular element $a \in A$ when π is applied n times?

$$a \mapsto \pi(a) \mapsto \pi^2(a) \mapsto \ldots \pi^n(a)$$

Those $n + 1$ elements of A cannot all be different!

$\Rightarrow \exists$ two different integers j, k, with $0 \leq j < k \leq n$, such that

$$\pi^k(a) = \pi^j(a).$$

Now apply π^{-j} on both sides ...

$$\pi^{k-j}(a) = a,$$

So we found ... an $m \in \{1, 2, \ldots, n\}$ s.t. $\pi^m(a) = a$. Namely $m = k - j$.

Some elements may return later than others

The **order** of an element, $o(a)$, is the smallest $m \geq 1$ s.t. $\pi^m(a) = a$.

On the previous slide we saw that $o(a)$ exists and $\leq n$. Different elements can have different orders.

Example

Let $\pi = (a\ b)(c\ d\ e)$. Then $o(a) = o(b) = 2$, but $o(c) = o(d) = o(e) = 3$.

There is some positive integer m which is divisible by all element orders. (For example, their product.)

Here the smallest such m is 6: π^6 returns each element to itself, that is, $\pi^6 = \iota$.

The **order** of a permutation, $o(\pi)$, is the smallest $m \geq 1$ s.t. $\pi^m = \iota$.

Every permutation returns to identity, sooner or later

Example

Consider the 26 English letters $A = \{a, b, c, \ldots, x, y, z\}$, and permutation

$$\pi = (abc)(defgh)(ijklmno)(pqrstuvwxyz).$$

The cycles have different lengths $3, 5, 7, 11$, and the smallest positive multiple of these numbers is

$$3 \cdot 5 \cdot 7 \cdot 11 = 1155,$$

so $o(\pi) = 1155$.

If you take a long text, and apply π repeatedly to all its letters (a "substitution cipher"), after 1155 repetitions you will certainly have your original text back!

Order and divisibility

Consider

- a set A, of cardinality n
- the set of all its permutations, of cardinality $n!$

We know that $o(a) \leq n$, but it could be basically any number between 1 and n. (Previous slide: $n = |A| = 26$; $o(a) = 3$, $o(q) = 11$.)

We know that $o(\pi)$ could be quite big, but it is finite. In fact $o(\pi) \mid n!$. (Proof: Consider the longest cycle of π ...)

Example

- Previous slide; $1155 \mid 26!$ (try in SageCell).
- Caesar cipher has order 26, which divides 26!
- ROT13 cipher has order 2, which divides 26!

A group of permutations, generated from one permutation

Given **one** permutation π (e.g. Caesar cipher), we can consider the group **generated by** π,

$$\langle \pi \rangle = \{\pi^n \ : \ n \in \mathbb{Z}\}.$$

This is **also** a group (associativity, neutral element, inverse!) but possibly much smaller than S_n. It is a **subgroup** of S_n.

Example

- By iterating the ROT13 cipher, we obtain only $26 \ll 26!$ different permutations.
- By iterating the Caesar cipher, we obtain only $26 \ll 26!$ different permutations.

A group of permutations, generated from several permutations

Consider a combination lock with positions $(a, b) \in P = \{0, 1, \ldots, 9\}^2$, and two "elementary" permutations:

- Rotating 1st dial, $f(a, b) = ((a + 1) \bmod 10,\ b)$
- Rotating 2nd dial, $g(a, b) = (a,\ (b + 1) \bmod 10)$

Generally, **any** permutation π of P rotates the first dial by some $+s$ positions, and the second dial by some $+t$ positions.

It is not difficult to see that $\pi = f^s\, g^t$, so every permutation **can be expressed** as a combination of these two "elementary" permutations.

The set of permutations obtainable from f and g is called the **group generated by f and g**, and written $\langle f, g \rangle$.

In **algebra**, there is much more to learn about groups, subgroups and generating, **but we stop here**.

Conjugates

- In any group G, two elements $\pi, \sigma \in G$ are *conjugates* if $\pi = \tau\sigma\tau^{-1}$ for some $\tau \in G$.
- The conjugate relation is an equivalence relation. (proof on blackboard)

Example

- (1234) and (1243) are conjugates in S_4, because

$$(1234) = (123)(1243)(132) = (123)(1243)(123)^{-1}.$$

Conjugates

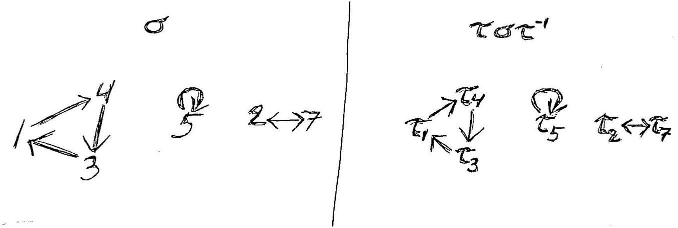
- If $\tau \in S_n$ is a permutation and $(a_1, \ldots, a_k)$ is a cycle, then

$$\tau(a_1 \ \ldots \ a_k)\tau^{-1} = (\tau(a_1) \ \cdots \ \tau(a_k)).$$

- If π and σ are conjugates, then they have the same number of cycles of length k.
- In the symmetric group S_n, the conjugate relation can thus be equivalently defined as follows:
  - $\pi, \sigma \in S_n$ are conjugates, if and only if they have equally many k-cycles for each $k = 1, \ldots, n$.

Conjugates

- The conjugates σ and $\tau\sigma\tau^{-1}$ in S_n have "the same structure", but the elements of the ground set $\{1, \ldots n\}$ are in different places in the cycles.



Conjugates

Example

The elements of S_4 are:

ι							
(12)	(13)	(14)	(23)	(24)	(34)		
(123)	(132)	(124)	(142)	(134)	(143)	(234)	(243)
(12)(34)	(13)(24)	(14)(23)					
(1234)	(1243)	(1324)	(1342)	(1423)	(1432)		

- The conjugate classes are the rows of this table.
- The group S_4 has five conjugate classes.
- How many conjugate classes does S_n have? There is no known closed formula (in terms of n).

Cycle notation

- A cycle (ab) of length 2 is called a *transposition*.

Theorem

Every permutation $\pi \in S_n$ can be written as the product of transpositions.

Proof.

- It is enough to show that every cycle $(a_1 \dots a_k)$ is the product of transpositions.
- $$(a_1 a_2 \dots, a_{k-1} a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_3)(a_1 a_2).$$

$\square$

Cycle notation

Theorem

Every permutation $\pi \in S_n$ can be written as the product of transpositions.

- The same permutation can be written as a product of transpositions in many different ways.

Example

$$(1234) = (12)(23)(34) = (14)(13)(12) = (12)(24)(23) = \dots .$$

Cycle notation

Theorem

1. *Every permutation* $\pi \in S_n$ *can be written as a product using the transpositions* $(1\ 2), (1\ 3), \dots, (1\ n)$.
2. *Every permutation* $\pi \in S_n$ *can be written as a product using the transpositions* $(1\ 2), (2\ 3), \dots, (n-1\ n)$.

Proof.

- It is enough to write every *transposition* as such a product.
- $(k\ \ell) = (1\ k)(1\ \ell)(1\ k)$. This proves 1.
-

$$(1\ k) = (k{-}1\ k)(k{-}2\ k{-}1)\cdots(2\,3)(1\,2)(2\,3)\cdots(k{-}2\ k{-}1)(k{-}1\ k).$$

This proves 2. □

Even and odd permutations

Theorem

For a permutation $\pi \in S_n$, its representations as a product of transpositions either all use an even number of transpositions, or they all use an odd number of transpositions.

- If $\pi \in S_n$ is the product of an even number transpositions, then we say that π is an *even* permutation, and that it has *sign* $\epsilon(\pi) = +1$.
- If $\pi \in S_n$ is the product of an odd number of transpositions, then we say that π is an *odd* permutation, and that it has *sign* $\epsilon(\pi) = -1$.

Even and odd permutations

Example

- A transposition

$$(j\ k) = (1\ j)(1\ k)(1\ j) = (1\ 3)(3\ j)(1\ 3)(1\ 2)(2\ k)(1\ 2)(1\ j) = \cdots$$

  is odd.
- The identity permutation $\iota = (j\ k)(j\ k)$ is even.
- The set of even permutations is denoted A_n.

Even and odd permutations

Example

- A cycle

$$(a_1, a_2, \ldots, a_{k-1} a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_3)(a_1 a_2)$$

  is even if its length k is odd, and it is odd if its length is even.
  (ANNOYING!)
- $\epsilon(\sigma\pi) = \epsilon(\sigma)\epsilon(\pi)$
  - even $\cdot$ even $=$ odd $\cdot$ odd $=$ even.
  - even $\cdot$ odd $=$ odd $\cdot$ even $=$ odd.
- So compositions of permutations is a map

$$A_n \times A_n \to A_n,$$

  and so the even permutations form a *subgroup* $A_n \subseteq S_n$. (the alternating group).

Even and odd permutations

Theorem

For a permutation $\pi \in S_n$, its representations as a product of transpositions either all use an even number of transpositions, or they all use an odd number of transpositions.

- For the proof, we need the following definition:

Definition

- An *inversion* in $\pi \in S_n$ is a pair $i < j$ such that $\pi_i > \pi_j$.
- $\operatorname{inv} \pi$ is the number of inversions in $\pi \in S_n$.

Example

The inversions in $13542 \in S_5$ are $(2,5)$, $(3,4)$, $(3,5)$, $(4,5)$.

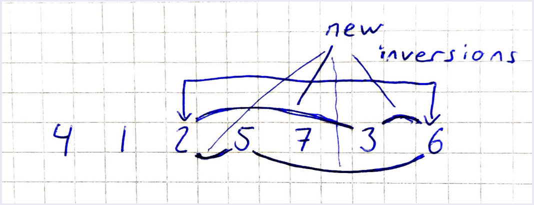
13542 13542 13542 13542

Even and odd permutations

Lemma

- *Let $\omega = (a\ b) \in S_n$ be a transposition, with $a < b$.*
- *Then $\operatorname{inv} \pi \circ \omega - \operatorname{inv} \pi$ is odd.*

Proof (illustration).



Even and odd permutations

Lemma

- *Let* $\omega = (a\ b) \in S_n$ *be a transposition, with* $a < b$*.*
- *Then* $\operatorname{inv} \pi \circ \omega - \operatorname{inv} \pi$ *is odd.*

Proof.

- If $i, j \notin \{a, b\}$, then $(i\ j)$ is an inversion in π if and only if it is an inversion in $\pi\omega$.
- If $a < i < b$ and either $\pi_i \leq \min(\pi_a, \pi_b)$ or $\pi_i \geq \max(\pi_a, \pi_b)$, then exactly one of the pairs (a, i) and (i, b) is an inversion, both in π and in $\pi\omega$.

Even and odd permutations

Lemma

- *Let* $\omega = (a\ b) \in S_n$ *be a transposition, with* $a < b$.
- *Then* $\operatorname{inv} \pi \circ \omega - \operatorname{inv} \pi$ *is odd.*

Proof (continued).

- Let $a < i < b$ and

$$\min(\pi_a, \pi_b) \leq \pi_i \leq \max(\pi_a, \pi_b).$$

- Then the pairs (a, i) and (i, b) are both inversions in one of the permutations (either in π or in $\pi\omega$), and in the other one neither of them is an inversion.

Even and odd permutations

Lemma

- *Let $\omega = (a\ b) \in S_n$ be a transposition, with $a < b$.*
- *Then $\operatorname{inv} \pi \circ \omega - \operatorname{inv} \pi$ is odd.*

Proof (continued).

- So the difference between the numbers of inversions

$$
\begin{aligned}
&|\{(i,j) : (i,j) \text{ inversion in } \pi \text{ but not in } \omega\pi, (i,j) \neq (a,b)\}| \\
-&|\{(i,j) : (i,j) \text{ inversion in } \omega\pi \text{ but not in } \pi, (i,j) \neq (a,b)\}|
\end{aligned}
$$

  is even.
- (a, b) is an inversion in either π or $\pi\omega$, and not in the other. $\square$

Even and odd permutations

Lemma

- $\operatorname{inv} \pi \circ \omega - \operatorname{inv} \pi$ *is an odd number if* ω *is a transposition*

Theorem

For a permutation $\pi \in S_n$*, its representations as a product of transpositions either all use an even number of transpositions, or they all use an odd number of transpositions.*

- By the lemma, if π is the product of an odd (even)number of transpositions, then $\operatorname{inv} \pi$ is odd (even).
- But the number of inversions is well defined.
- So the parity of the permutation is also well defined.

Fixed points of permutations

Example

- Each of n guests have brought gifts to a party, and these guests should be redistributed among the guests.
- Let $r(x)$ be the guest that gets the gift brought by x.
- We want

  $$r : \{\text{Guests}\} \to \{\text{Guests}\}$$

  to be surjectve (everyone should get a gift).
- We want $r(x) \neq x$ for all x (nobody should get back the same gift that they brought to the party).
- In how many ways can we redistribute the gifts with these rules?

Fixed points of permutations

- Recall that a permutation is a bijection $X \to X$.
- The set of permutations of $X = \{1, \ldots, n\}$ is the *symmetric group* S_n.
- A *fixed point* of $\pi \in S_n$ is an element $x \in X$ such that $\pi(x) = x$.
- A permutation that has no fixed points is called a *derangement*.
- How many derangements are there in S_n?

Fixed points of permutations

- Use the inclusion exclusion principle.
- For $i \in X$, let $A_i = \{\pi \in S_n : \pi(i) = i\}$.
- The number of permutations with k prescribed fixed points is

$$|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!,$$

  because the $n-k$ other elements must be permuted internally.
- For $k = 1, \ldots, n$,

$$s_k = \sum_{|B|=k} \left| \bigcap_{i \in B} A_i \right| = \binom{n}{k}(n-k)! = \frac{n!}{k!}.$$

Fixed points of permutations

- The number of non-derangements is

$$\begin{aligned} |A_i \cup \cdots \cup A_n| &= \sum_{k=1}^{n} (-1)^{k-1} s_k \\ &= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} \end{aligned}$$

- So the number of derangements is

$$\begin{aligned} n! - |A_i \cup \cdots \cup A_n| &= \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!} \\ &= n! \sum_{k=0}^{n} (-1)^{k} \frac{1}{k!} \end{aligned}$$

Fixed points of permutations

- Fact from Calculus 1:
$$\sum_{k=0}^{\infty} t^k \frac{1}{k!} = \mathrm{e}^t.$$
- So the number of derangements of n elements is
$$D_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} = n!\mathrm{e}^{-1} - \sum_{k=n+1}^{\infty} (-1)^k \frac{n!}{k!}.$$
- $$\left| D_n - \frac{n!}{\mathrm{e}} \right| = \left| \sum_{k=n+1}^{\infty} (-1)^k \frac{n!}{k!} \right| \le \frac{n!}{(n+1)!} = \frac{1}{n+1} < \frac{1}{2}$$
- So D_n is the closest integer to $n!/\mathrm{e}$.

Fixed points of permutations

Example

- Each of n guests have brought gifts to a party, and put them in a pile on a table.
- Secret Santa comes and gives a (uniformly) random gift from the table to each guest.
- The probability that no guest gets her own gift back is (very very close to)

$$1/\mathrm{e} \approx 0.368$$

regardless of the number of guests!

Part 3: Graph theory

Motivation I

"...networks may be used to model a huge array of phenomena across all scientific and social disciplines. Examples include the World Wide Web, citation networks, social networks (e.g., Facebook), recommendation networks (e.g., Netflix), gene regulatory networks, neural connectivity networks, oscillator networks, sports playoff networks, road and traffic networks, chemical networks, economic networks, epidemiological networks, game theory, geospatial networks, metabolic networks, protein networks and food webs, to name a few."

(Grady & Polimeni, Discrete Calculus, Springer 2010.)

This course: A small glimpse of graph theory

Graph theory is a broad topic, big enough for several university-level courses. We are touching the topic in one week.

Compare:

- Computer science courses: focus often in algorithms (e.g. routing, finding spanning trees etc.)
- Our focus: graphs as mathematical objects, using tools of discrete mathematics — functions, bijections, matrices . . .

Graph

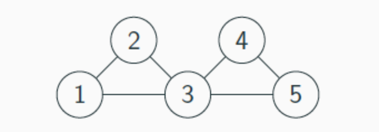
A **graph** is a pair (V, E), where

- V is a set of **vertices** (also called nodes, points).
- E is some set of two-element sets $\{u, v\}$, where $u, v \in V$ and $u \neq v$. These are called **edges**, arcs or links.

Vertices connected by an edge are called **neighbors**.

Example

- $V = \{1, 2, 3, 4, 5\}$
- $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$



Variations

Many variants of the idea, applicable in different situations.

We might allow ...

- ... one-vertex edges $\{u\}$, understood as a connection from a vertex to itself (graph with *loops*)
- ... multiple edges between two vertices (*multigraph*)
- ... directed edges: ordered pairs (u, v), understood as a connection **from** u **to** v — *directed graph* or *digraph*

Caveat. Terminology varies. Sometimes people use *simple graph* to rule out loops, and *undirected graph* to say that edges are undirected (sets) instead of directed (pairs).

Motivation II

Graphs can represent many things, both in math and in real world.

Concrete locations and physical connections.

- cities and road network
- islands and bridges (Bridges of Königsberg)
- electrical components and wires

More abstractly, **states** of some process and **transitions**

- money in wallet and wins/losses (gambling, stock market)
- games and movements/plays (chess, go, ...)
- chessboard squares and knight movements (Knight's tour)

More abstractly, "some relation" between things

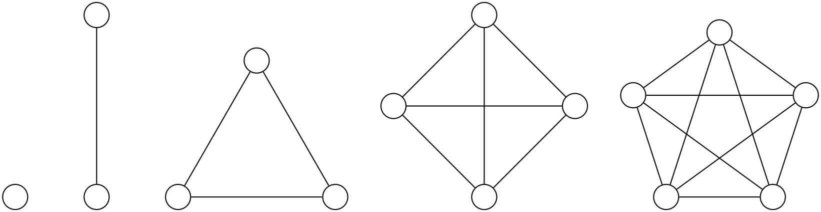
- V=people, E="have met"
- V = people $\cup$ articles, $(x, y) \in E$ if x is an author of y

Some examples: Complete graph

Complete graph (or **clique**) K_n:

- n vertices, e.g. $V = \{1, 2, \ldots, n\}$
- Every pair $u \neq v$ has an edge, so $\binom{n}{2} = n(n-1)/2$ edges

Check: What are K_1, K_2, K_3, K_4, K_5?



Some more examples

- **Path graph** P_n ($n \geq 1$) has n consecutive vertices, e.g. $\{1, \ldots, n\}$: with edges $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$
- **Cycle graph** C_n ($n \geq 3$) similar, but one more edge $\{n, 1\}$
  C_3 called "triangle", C_4 called "square"
- **Star graph** S_n has a central vertex, and n others connected to it ($n+1$ vertices total)
- **Empty graph** has n vertices but no edges!
- Vertices (corners) and edges of a **polyhedron**, such as **cube**.
- Also many other named graphs, but these are the most common.
- Different names can refer to the same structure (e.g. K_2 and P_2 and S_1), more about structural similarity later.

Paths, distance, connected

We often think that two vertices, even if not neighbors, can be "indirectly connected" via other vertices.

A **path** is an ordered sequence of vertices

$$(v_0, v_1, \ldots, v_n)$$

such that any two consecutive vertices v_i and v_{i+1} are neighbors.

This path has **length** n, the number of "steps" (edges). Note that it has $n+1$ vertices (beware of fencepost).

This path is **from** v_0 **to** v_n, and it **connects** those two vertices.

A graph is **connected** if every pair of vertices has some path that connects them. Otherwise the graph is **disconnected**. Examples: Empty graph, disjoint cycles, . . .

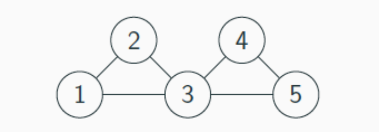
The **distance** or **path length** between u and v is the length of the *shortest* path that connects them. (Examples on blackboard: Cycle graph, cube)

Degree

The **degree** $d(v)$ of a node v is the number of edges that have v as one of their endpoints.

Example

In the graph below, $d(1) = d(2) = d(4) = d(5) = 2$, and $d(3) = 4$.



Handshaking lemma

Theorem

In any graph we have

$$2 \cdot |E| = \sum_{v \in V} d(v).$$

Proof.

The sum counts each edge $\{u, v\}$ twice: once in the degree of u, and once in the edgree of v. □

This simple fact is sometimes very useful. Example: Can we create a graph that has an odd number of odd-degree vertices? (No, because the sum of degrees is always even.)

Regular graph

A graph is ***k*-regular** if all vertices have degree k.

Example

- complete graph K_n is $(n-1)$-regular
- cycle graph C_n is 2-regular
- empty graph of n vertices is 0-regular
- path graph P_n is **not regular** when $n \geq 3$:
  The endpoints have degree 1, others have degree 2.
  But note boundary cases P_1, P_2.

Handshaking lemma $\Rightarrow$ An odd-regular graph cannot have an odd number of vertices. (Consider $k = 1$, $k = 3$)

Subgraph

If $G = (V, E)$ and $G' = (V', E')$ are graphs, G' is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$.

I.e. take *some* of the original graph's vertices, and *some* of the edges between them (perhaps all).

Existence/Nonexistence of certain kinds of subgraphs is important in applications. E.g. a graph is **acyclic** if it does not contain any cycle.

Example

1. Cycloalkanes are hydrocarbons that contain **one cycle**; typically different chemical properties than acyclic hydrocarbons
2. In computer science, genetics etc. we often study **trees**, which are **acyclic connected graphs**.
3. Finding subgraphs that are **cliques** (K_n) is important e.g. in social networks, graph coloring etc.

Vertex coloring

Definition

- A **(vertex) k-coloring** of the graph $G = (V, E)$ is a function

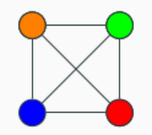
$$\gamma : V \to \{1, 2, \ldots, k\}$$

  such that

$$\text{if } \{u, v\} \in E \text{ then } \gamma(u) \neq \gamma(v).$$

- The **chromatic number** $\chi(G)$ is the smallest number k such that there is a k-coloring of G.

Often $\{1, 2, \ldots k\}$ called "colors". Here is a 4-coloring of K_4.



Chromatic number examples

- $\chi(K_n) = n$
- $\chi(P_n) = 2$ (for $n \geq 2$). Proof: alternating colors
- $\chi(C_n) = 2$ if n even, but 3 if n odd (!)

Subgraphs and coloring

Theorem

If H is a subgraph of G, then $\chi(G) \geq \chi(H)$.

In particular, if G contains a k-clique (complete k-element subgraph), then $\chi(G) \geq k$.

The **clique number** $\omega(G)$ is the size of the **largest** clique in G. So **lower bound**

$$\chi(G) \geq \omega(G)$$

Generally, subgraphs (cliques or others) are often useful for proving lower bounds on $\chi(G)$ ("at least this many colors needed").

Coloring and partition

A k-coloring is equivalent to **partitioning** the vertices into k parts, such that there are **no edges inside any part**.
$G = (V, E)$ is **bipartite**, if we can partition $V = V_1 \cup V_2$ so that all edges are between the parts. Equivalently, this is a 2-coloring.

Sometimes we know a partition from the outset because we have two different *kinds* of vertices, and the graph represents a **relation** between the two parts.

Example

- vertices: people $\cup$ books, edge = "x is an author of y"
- vertices: bus lines $\cup$ stops, edge = "x stops at y"

Sometimes we *don't* know a partition (or coloring), and finding it is the task.

Conflict graphs

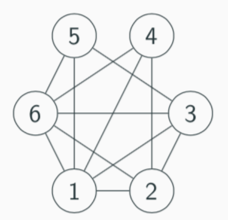
Example

- Six students Alice, Bob, Camilla, David, Erika, Fred are doing six different projects in the following groups:
  1. A,B,C,F
  2. B,D,E
  3. C,F
  4. B,E
  5. A,C,F
  6. D,E,F
- Each project requires one day to complete, which the participants have to spend together. In how many days can all the projects be completed?

Conflict graphs

Example (Continued)

- Construct the *conflict graph*, $G = (V, E)$ whose nodes are the tasks, and whose edges represent pairs of tasks that can not be completed on the same day.

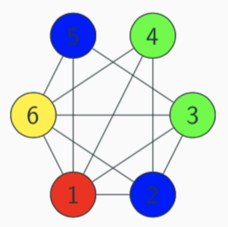


- If $\gamma : V \to \{1, \ldots, k\}$ is a graph coloring, then we can complete each task v on day number $\gamma(v)$.
- So the smallest number of days needed is $\chi(G)$.

Conflict graphs

Example (Continued)

- We can color the graph with 4 colors as below, so $\chi(G) \leq 4$.



- On the other hand, the nodes $\{1, 2, 3, 6\}$ are pairwise connected, so need four different colors.
- Thus, $\chi(G) = 4$.

Greedy algorithm

- Finding the chromatic number of a graph is a difficult problem.
- There is no known algorithm whose complexity grows polynomially with the number of vertices.
- Any known coloring gives an upper bound of $\chi(G)$.
- The following **greedy algorithm** often gives useful **upper bounds** ("this many colors is enough").
- Requires an ordering $\{v_1, \ldots, v_n\}$ of the vertices of V.
- The number of colors needed may depend on the ordering.

Greedy algorithm

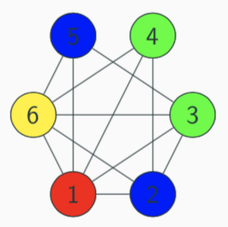
- Let $V = \{v_1, \ldots, v_n\}$.
- Let $\gamma(v_1) = 1$
- If $v_1, \ldots, v_{k-1}$ have already been colored, let

$$\gamma(v_k) = \min\{i \geq 1 : \gamma(v_j) \neq i \text{ for all } j < k \text{ for which } \{v_j, v_k\} \in E\}.$$

Greedy algorithm

Example

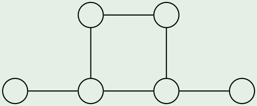
- Color the previous conflict graph with the greedy algorithm.
- The vertices are already labelled $1, \ldots 6$.
- Visualize the "colors" $1, 2, 3, 4$ as red, blue, green, yellow, in that order.



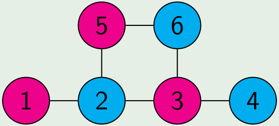
Greedy algorithm

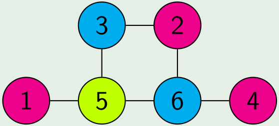
Example

- Color the following graph with the greedy algorithm.



- Depending on how you order the nodes, you need either two or three colors.





Greedy algorithm

Theorem

- *Let* $G = (V, E)$ *be a graph with* $\chi(G) = k$.
- *Then there exists an ordering* $v_1, v_2, \ldots, v_n$ *of the vertices such that the greedy algorithm colors the graph with* k *colors, if coloring the vertices in this order.*

- So if we can perform the greedy algorithm for all possible orderings of V, we can compute the chromatic number *exactly*.
- But there are $n!$ possible ways to order V, so this is not an efficient algorithm.

Greedy algorithm

Sketch of proof.

- Let $\gamma : V \to \{1, 2, \ldots, k\}$ be some coloring of G with $\chi(G) = k$ colors.
- Let $V_i \subseteq V$ be the set of vertices with $\gamma(v) = i$. So there are no edges between two nodes in V_i.
- Order the vertices such that all nodes in V_1 come first, then all nodes in V_2, and so on.
- Let $\delta : V \to \{1, 2, \ldots, k\}$ be a greedy graph coloring with respect to this ordering.
- By induction: $\delta(v) \leq i$ for all $v \in V_i$.
- So the greedy algorithm colors $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with k colors. $\square$

Maximum degree gives upper bound

Theorem

- *Let G be a graph, where all nodes have degree $\leq d$.*
- *Then $\chi(G) \leq d + 1$.*

Proof.

- Order the vertices arbitrarily, and color the graph using the greedy algorithm.
- For each vertex v_k, the set $\{v_j : j < k, \{j, k\} \in E\}$ has size $\leq d$, so at most d colors are used for those vertices.
- So v_k can be colored with at least one of the colors $1, 2, \ldots, d + 1$.
- So the greedy algorithm requires at most $d + 1$ colors, so $\chi(G) \leq d + 1$. □

Greedy algorithm

Theorem

- *Let G be a graph, where all nodes have degree $\leq d$.*
- *Then $\chi(G) \leq d + 1$.*

Theorem (Brooks' Theorem, 1941)

- *Let G be a graph, where all nodes have degree $\leq d$.*
- *If $\chi(G) = d + 1$, then G is either a complete graph K_n or an odd cycle.*

Bounds could be very loose

Example

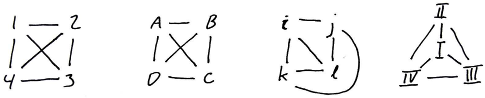
Star graph S_n has max degree n, giving upper bound $n+1$, but in reality $\chi = 2$

Example

Mycielski graphs have clique number 2 (no triangles at all!), giving lower bound 2, but in reality χ can be arbitrarily large

Isomorphism

When do two graph have “the same structure”?



The four graphs above look different, still they are all “complete on 4 vertices”, and share the “same structure”.

The following definition makes the notion precise, so that we can (hopefully) *prove* that two graphs have the same or different structures.

Isomorphism

Definition

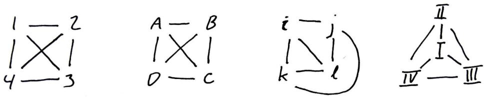
An **isomorphism** between two graphs $G = (V, E)$ and $G' = (V', E')$ is a

- **bijection** $f : V \to V'$
- such that **all neighborhood relations are identical**:

$$\forall u, v \in V : \ \{u, v\} \in E \longleftrightarrow \{f(u), f(v)\} \in E'.$$

If there **exists** an isomorphism between G and G', we say the graphs are **isomorphic** and write $G \simeq G'$.

These four graphs are all isomorphic.



Being isomorphic is a strong property

If $G \simeq G'$, this **implies** many things:

- $|V| = |V'|$, otherwise no bijection at all!
- Every vertex v maps to a vertex $f(v)$ *of the same degree*.
- Corollary: For each possible degree, G and G' have the *same number* of vertices of that degree
- Corollary: $|E| = |E'|$ (same total number of edges)
- Every subgraph of G maps to a subgraph of G' with the same structure
  - every 3-cycle maps to a 3-cycle
  - every K_4 maps to a K_4

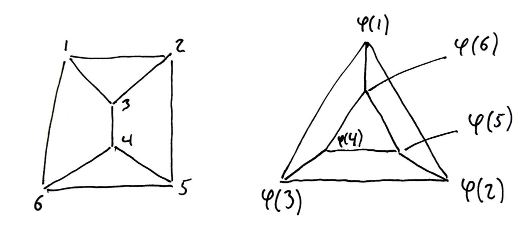
These are often helpful in proving that two graphs are **not** isomorphic: e.g. if G has two vertices of degree 4, but G' has only one such vertex, then $G \not\simeq G'$.

But remember how implication works. Having the same number of vertices, edges etc. is not a proof of isomorphism. For a conclusive proof, **construct** an isomorphism!

Isomorphism examples

Example

- Two complete graphs of same size are isomorphic.
- Two path graphs of same length are isomorphic.
- Two cycle graphs of same length are isomorphic.
- The graphs below are isomorphic; φ is an isomorphism.



Non-isomorphism examples

Example

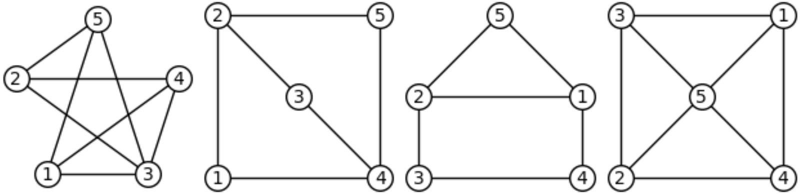
- P_n and C_n have same number of vertices, but number of edges is enough to see they are not isomorphic.
- Adding a diagonal to C_6 in two ways: Both have $|E| = 7$, *and* same number of vertices of each degree, yet nonisomorphic
- 6-cycle vs. union of two disjoint 3-cycles: Each graph has 6 edges, and same number of vertices of each degree

If easy methods fail to show nonisomorphism, we simply need to *prove*, by whatever means, that no bijection between the vertices can be an isomorphism. This *could* be difficult.

The last resort would be to *try* all possible $n!$ bijections and *test* if one of them is an isomorphism! (An extreme "proof by cases".)

Deciding isomorphism could be hard

Which of the following four graphs are isomorphic and which are not? Why?



Brute force method: We *could* just try all $5! = 120$ bijections and check if neighborhoods are preserved.

Saving work: If φ is an isomorphism, then $d(v) = d(\phi(v))$, which severely *restricts* which vertices can be mapped where.

Algorithmic complexity

It is currently not known exactly *how* difficult it is to determine (by a computer program) if two graphs are isomorphic. See Graph isomorphism problem.

However, practical algorithms and computer programs exist for very large graphs.

Isomorphism is an equivalence

Theorem

The isomorphism relation $\simeq$ is an equivalence.

Proof.

- Reflexivity: $G \simeq G$ by identity function
- Symmetry: if there is an isomorphism $f : G \to G'$, then its inverse function is an isomorphism $f^{-1} : G' \to G$.
- Transitivity: If $f_1 : G \to G'$ and $f_2 : G' \to G''$ are isomorphisms, then so is $(f_2 \circ f_1) : G \to G''$.

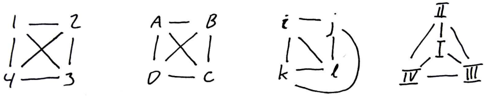
$\square$

Isomorphism classes

Recall that an equivalence relation groups objects into **equivalence classes**. Here a class contains all graphs that have "the same structure".

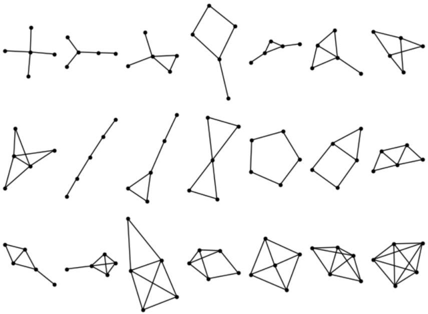
Our earlier examples K_n (complete), C_n (cycle), P_n (path) and so on, are not in fact "graphs", but **descriptions of graph structure**.

The vertices of a complete graph K_4 could be named, or **labeled** in many ways, giving *different but isomorphic* graphs. We can say that each of these graphs is "**a** K_4" (with indefinite article).



Unlabeled graph examples

Here are *all* unlabeled connected graphs of 5 vertices, that is, all such graph *structures*.



Each could be *labeled* in several ways.

How many graphs exist?

Let the vertices be $V = \{1, 2, \ldots, n\}$. How many (a) graphs, (b) unlabeled graphs, (c) unlabeled connected graphs exist?

(a) Easy: $2^{\binom{n}{2}} = 2^{k(k-1)/2}$, because $\binom{n}{2}$ possible edges, see A006125

(b,c) Harder, only known up to $n = 19$, see A000088 and A001349

n	(a) graphs	(b) unlab. graphs	(c) unlab. conn. graphs
2	$2^1 =$ 2	2	1
3	$2^3 =$ 8	4	2
4	$2^6 =$ 64	11	6
5	$2^{10} =$ 1 024	34	21
6	$2^{15} =$ 32 768	156	112
7	$2^{21} =$ 2 097 152	1 044	853

Demo: `https://sagecell.sagemath.org/?q=mweiqo`

Adjacency matrix

- Let $G = (V, E)$ be a graph, and $V = \{v_1, \ldots, v_n\}$.
- The *adjacency matrix* of G is the $n \times n$ matrix A with

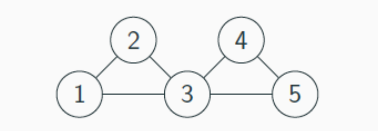
$$A(j,k) = \begin{cases} 1 & \text{if } \{v_j, v_k\} \in E \\ 0 & \text{otherwise} \end{cases}$$

- So the adjacency matrix has an entry 1 in the i^{th} row and j^{th} column if the v_i and v_j are neighbours.

Adjacency matrix

Example

- The adjacency matrix of the graph



is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency matrix

- As in Matrix Algebra, the product of two $n \times n$ matrices A and B is the $n \times n$ matrix AB with

$$AB(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j).$$

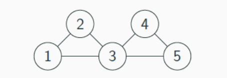
- In other words, $AB(i,j)$ is the *scalar product* of the i^{th} row of A and the j^{th} column of B.
- The product of adjacency matrices can be interpreted combinatorially.

Adjacency matrix

Theorem

- *Let A be the adjacency matrix of the graph G, with nodes $v_1, \ldots, v_n$.*
- *Then $A^k(i,j)$ is the number of paths of length k from v_i to v_j in G, for $k \in \mathbb{N}$.*

Example



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 2 & 3 & 5 & 2 & 2 \\ 3 & 2 & 5 & 2 & 2 \\ 5 & 5 & 4 & 5 & 5 \\ 2 & 2 & 5 & 2 & 3 \\ 2 & 2 & 5 & 3 & 2 \end{pmatrix}$$

- The entry $A^3(2,3) = 5$ tells us that there are five paths of length 3 from node 2 to node 3.

Adjacency matrix

Theorem

- *Let A be the adjacency matrix of the graph G, with nodes $v_1, \ldots, v_n$.*
- *Then $A^k(i,j)$ is the number of paths of length k from v_i to v_j in G, for $k \in \mathbb{N}$.*

Proof.

- By induction:
- Base case $n = 0$: A^0 is the identity matrix $A^0 = I_n$, with

$$I_n(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The only paths of length 0 in G go from a node v_i to itself, so the number of such paths is $I_n(i,j)$.

Adjacency matrix

Proof (Continued).

- Induction step: Assume $A^m(i,j)$ is the number of paths of length m from v_i to v_j in G.
- A path of length $m+1$ in G from v_i to v_j is a path of length m from v_i to some node v_ℓ, together with an edge from v_ℓ to v_j.
- So the number of such paths is

$$\sum_{\substack{\ell\in\{1,\dots n\}\\ \{v_\ell,v_j\}\in E}} A^m(i,\ell) = \sum_{\substack{\ell\in\{1,\dots n\}\\ A(\ell,j)=1}} A^m(i,\ell) = \sum_{\ell=1}^{n} A^m(i,\ell)A(\ell,j) = A^{m+1}(i,j).$$

- By the induction principle, $A^k(i,j)$ is the number of paths of length k from v_i to v_j in G, for all $k\in\mathbb{N}$. $\square$

Powers of adjacency matrix

SageMath demo https://sagecell.sagemath.org/?q=kntrgg

Note (connection to stochastics).
Very similar matrix powers appear with the *transition matrices* of Markov chains.

Course: MS-C2111 Stochastic processes.

- A system has n states, and at each time it is in exactly one state.
- $n \times n$ transition matrix, whose element A_{ij} indicates the *probability* of moving from state i to state j.
- Here the matrix elements are probabilities, not zeros and ones.
- Same idea: A^m tells what happens when we perform m consecutive transitions.

Planar graphs

A graph is **planar** if it *can* be drawn on plane, without any edges crossing.

- e.g. K_4, C_n, S_n are planar
- e.g. K_5 is not planar

Many practical applications, but here we consider a less practical one: Map coloring.

A planar map of countries can be *transformed* into a planar graph. (Why? BLACKBOARD)

Coloring planar graphs

Question: How many colors are enough to color *any* planar map of countries?
It is easy to see that three are not enough (there can be four countries all neighboring each other.)

- Six is enough — relatively simple proof exists.
- Heawood (1890): Five is enough. A proof of a couple of pages.
- Appel & Haken (1976): Four is enough. Computer-assisted proof by cases (1 834 cases!).

Six-color theorem — Ingredients

To prove that any planar graph can be six-colored, these are the ingredients:

- Handshaking lemma: $2|E| = \sum_v d(v)$, true for any graph
- Euler characteristic: $|V| - |E| + |F| = 2$ in any planar graph (F are the "faces", the areas surrounded by edges)
- Every planar graph contains a vertex with degree ≤ 5 (From combining the previous two claims)
- Induction on number of vertices

We may (time permitting) do some of these on the blackboard.

Part 4: Number theory

Number theory

Number theory means the theory of *integers*.

Restricting to integers makes some things easier (or more concrete), but some others harder (or at least different).

- Compare solving $3x + 5y = 1$ for x, y in reals vs. in integers!

Nowadays number theory has lots of applications in computing (algorithmics, coding theory, cryptography).

Some early views of number theory . . .

- Pythagoras of Samos (c. 570–495 BC):
  Everything is made of "numbers" (integers), e.g. their ratios
  → Modern view: Kind of, but you need more than just ratios
- Carl Gauss (1777–1855):
  Mathematics is the queen of sciences, and number theory is the queen of mathematics
- Leopold Kronecker (1823–1891):
  God made the integers, all else is the work of man
  → Modern view: Also integers can be constructed from more elementary things
- G.H. Hardy (1877–1947):
  Number theory is an honest branch of math because it has no applications (e.g. to war).
  → Soon proved wrong

Caveat: These are paraphrases, not exact quotes from these people

Divisibility

- A number $n \in \mathbb{Z}$ is **divisible** by $m \in \mathbb{Z}$ if there exists $k \in \mathbb{Z}$ such that

$$mk = n.$$

- Then we also say that m **divides** n, or in formulas $m \mid n$.
- Or, m is **a divisor** of n, or n is **a multiple** of m
- Negation ("not divisible") written $m \nmid n$.

Example

- $2 \mid 4$.
- $6 \mid 12$
- $6 \nmid 9$
- $0 \nmid n$ when $n \neq 0$.
- $1 \mid n$ when $n \in \mathbb{Z}$.
- $n \mid 0$ when $n \in \mathbb{Z}$.
- $n \nmid 1$ when $n \neq \pm 1$.

Practical factoring

How do we *find* whether $m \mid n$? Typically we perform the division (long division on paper, or calculator) and check if the result is an integer.

For some divisors we have handy rules (when the numbers are presented in the usual ten-based positional notation).

An integer is divisible by ...

- 2, iff its last digit is likewise (i.e. is $0, 2, 4, 6, 8$)
- 5, iff its last digit is likewise (i.e. is 0 or 5)
- 10, iff its last digit is likewise (i.e. is 0)
- 3, iff its *sum of digits* is likewise divisible by 3
- 9, iff its *sum of digits* is likewise divisible by 9

Iff is math slang for "if and only if"

Why do these work? All easily proven via congruences (next lecture)

Primes and factorization

An integer $p \geq 2$ is **prime** if its only positive divisors are 1 and p.

Examples: $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

(Fun fact: There are infinitely many primes.)

Contrariwise — if a number n is *not* prime, it can be factored as $m = ab$ where $1 < a \leq b < n$.

How to find such factorization? Naive method (good for small numbers):

- **Try** dividing by all primes $2 \leq p \leq \sqrt{n}$.
- Why is $\sqrt{n}$ enough, to find a factor if there is any? See Ex. 6a6j.

Prime factorization

If $n = ab$ and a or b is not prime, we can continue factoring *them*. Finally we get a **prime factorization**

$$n = p_1 p_2 \cdots p_k$$

where some primes may appear multiple times, or

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$$

if we collect multiple occurrences of each prime into a power.

Useful facts:

- Every integer $n \geq 2$ has a prime factorization (possibly just one factor, if n itself is prime)
- It is unique (up to order)

Divisibility

If $m \mid n_1$ and $m \mid n_2$, then $m \mid (a_1 n_1 + a_2 n_2)$ for all integers a_1, a_2.

(Cf. exercise 6A6)

Example

Since $3 \mid 9$ and $3 \mid 15$, it follows that $3 \mid 4 \cdot 15 - 2 \cdot 9 = 42$.

Divisibility

- So the set of common divisors of n_1 and n_2 is the same as the set of common divisors of n_2 and $n_1 - an_2$.
- In particular, the **greatest common divisor** satisfies

$$\gcd(n_1, n_2) = \gcd(n_1 - an_2, n_2) \text{ for all } a.$$

Example

$$\begin{aligned} \gcd(162, 114) &= \gcd(48, 114) & &= \gcd(48, 18) \\ &= \gcd(12, 18) & &= \gcd(12, 6) \\ &= \gcd(6, 6) & &= 6. \end{aligned}$$

- This illustrates the *Euclidean algorithm* for computing the greatest common divisor of two numbers.

Euclidean division

Theorem (Euclidean division)

- *Let $a, b \in \mathbb{Z}$, with $b > 0$.*
- *Then there exist unique numbers $q, r \in \mathbb{Z}$ with $0 \leq r < b$ and*

$$a = qb + r.$$

- q is the **quotient** of a when divided by b. (In programming languages often called **integer division**)
- r is the **modulus** or **remainder** of a, when divided by b. Written

$$a \bmod b$$

  (In programming languages often as % operator)
- So $\frac{a}{b} = q + \frac{r}{b}$.

Euclidean division

Example

- When dividing $a = 19$ by $b = 7$, the quotient is $q = 2$ and the remainder is $r = 5$.
- When dividing $a = -19$ by $b = 7$, the quotient is $q = -3$ and the remainder is $r = 2$.

- The proof of Euclidean division is simple but tedious.
- Idea: r is the smallest non-negative number in $S\{a - kb : k \in \mathbb{Z}\}$.
- Show that this r is the only element in S with $0 \leq r < b$.

Euclidean algorithm

- Let $r = a - qb$ be the remainder of a modulo b.
- Then $\gcd(a, b) = \gcd(r, b) = \gcd(b, r)$.
- $\gcd(b, 0) = b$ for all integers $b \neq 0$.
- This gives an *algorithm* for computing the greatest common divisor

$$\gcd(a, b)$$

  of two numbers $a \geq b$ in $O(\log a)$ steps.

Euclidean algorithm

Example

- To compute $\gcd(162, 114)$:

$$\begin{aligned} 162 &= 1 \cdot 114 + 48 \\ 114 &= 2 \cdot 48 + 18 \\ 48 &= 2 \cdot 18 + 12 \\ 18 &= 1 \cdot 12 + 6 \\ 12 &= 2 \cdot 6 + 0 \end{aligned}$$

- The greatest common divisor is the last non-zero remainder:

$$\gcd(162, 114) = 6.$$

Extended Euclidean algorithm

- In each iteration of the Euclidean algorithm, the remainder is written as an integer combination of previous remianders:

Example

$$
\begin{aligned}
48 &= 162 - 1 \cdot 114 \\
18 &= 114 - 2 \cdot 48 \\
12 &= 48 - 2 \cdot 18 \\
6 &= 18 - 1 \cdot 12
\end{aligned}
$$

- This can be used to write the final remainder $\gcd(a, b)$ as an integer combination $xa + yb$, where $x, y \in \mathbb{Z}$.

Extended Euclidean algorithm

Example

$$\begin{aligned} 48 &= 162 - 1 \cdot 114 \\ 18 &= 114 - 2 \cdot 48 \\ 12 &= 48 - 2 \cdot 18 \\ 6 &= 18 - 1 \cdot 12 \end{aligned}$$

- We use this to write $6 = \gcd(114, 162)$ as an integer combination

$$114x + 162y, \quad \text{where } x, y \in \mathbb{Z}.$$

$$\begin{aligned} 6 &= 18 - 12 \\ &= 18 - (48 - 2 \cdot 18) && = 3 \cdot 18 - 48 \\ &= 3(114 - 2 \cdot 48) - 48 && = 3 \cdot 114 - 7 \cdot 48 \\ &= 3 \cdot 114 - 7(162 - 114) && = 10 \cdot 114 - 7 \cdot 162. \end{aligned}$$

Linear Diophantine equations in two variables

- An equation where the variables are integer valued is called a *Diophantine* equation.
- The extended Euclidean algorithm gives a solution (x_B, y_B) to the Diophantine equation

$$\gcd(a, b) = ax + by.$$

- The integers (x_B, y_B) are the *Bézout coefficients* of a and b.

Linear Diophantine equations in two variables

- $$\gcd(a, b) = ax_B + by_B.$$
- If $\gcd(a, b) \mid c$, then the pair

$$(x_0, y_0) = \frac{c}{\gcd(a, b)}(x_B, y_B)$$

is an integer solution to the equation $c = ax + by$.

Linear Diophantine equations in two variables

- If $\gcd(a, b) \nmid c$, can there still be integer solutions to the equation

$$c = ax + by?$$

- No! Because $\gcd(a, b) \mid ax + by$ for all integers x, y.

Linear Diophantine equations in two variables

Theorem

- *The Diophantine equation*

$$c = ax + by$$

has integer solutions if and only if $\gcd(a, b) \mid c$.

- *If* $\gcd(a, b) \mid c$, *then one* particular *solution* (x_0, y_0) *is given by Euclid's extended algorithm.*
- *Let* $a' = \frac{a}{\gcd(a,b)}$ *and* $b' = \frac{b}{\gcd(a,b)}$.
- *Then* all *integer solutions to the equation are*

$$(x_0 + nb', y_0 - na'), \; n \in \mathbb{Z}.$$

- To prove this, we first must address the issue of *unique factorization*.

Dividing a product

Lemma

if $\gcd(a, b) = 1$ *and* $a \mid bc$*, then* $a \mid c$.

- If $\gcd(a, b) = 1$, then $1 = xa + yb$ holds for some $x, y \in \mathbb{Z}$, so

$$c = xca + ybc.$$

- Since a divides

$$xca + ybc$$

, it also divides c.

Unique factorization

- So if p is a prime (only divisible by 1 and itself) such that $p \mid bc$, then either $p \mid b$ or $p \mid c$.
- It follows that every number can be written as a product of primes *in a unique way*.
-
$$210 = 7 \cdot 30 = 10 \cdot 21 = 6 \cdot 35 = \cdots = 2 \cdot 3 \cdot 5 \cdot 7$$
can not be written as a product of *primes* in any other way.

Unique factorization

- We want to divide a large number N into prime factors
- First, we find a prime p that divides N.
- Then we factorize the smaller number N/p.

Example

$$\begin{aligned} 10452 &= 2 \cdot 5226 \\ &= 2^2 \cdot 2613 \\ &= 2^2 \cdot 3 \cdot 871 \\ &= 2^2 \cdot 3 \cdot 13 \cdot 67. \end{aligned}$$

- We see that 67 is a prime, because it is not divisible by any prime $\leq \sqrt{67} < 9$.

Linear Diophantine equations in two variables

- We are now ready to prove the following theorem.

Theorem

- *The Diophantine equation*

$$c = ax + by$$

has integer solutions if and only if $\gcd(a, b) \mid c$.

- *If* $\gcd(a, b) \mid c$, *then one* particular *solution* (x_0, y_0) *is given by Euclid's extended algorithm.*
- *Let* $a' = \frac{a}{\gcd(a,b)}$ *and* $b' = \frac{b}{\gcd(a,b)}$.
- *Then all integer solutions to the equation are*

$$(x_0 + nb', y_0 - na')\,, \;\; n \in \mathbb{Z}.$$

Linear Diophantine equations in two variables

Proof.

- $$a' = \frac{a}{\gcd(a, b)} \text{ and } b' = \frac{b}{\gcd(a, b)}.$$
- $$\begin{aligned} a(x_0 + nb') + b(y_0 - na') &= ax_0 + by_0 + (nab' - nba') \\ &= c + 0, \end{aligned}$$

so $(x_0 + nb', y_0 - na')$ is a solution.

Linear Diophantine equations in two variables

Proof (Continued).

- If (x, y) is an arbitrary solution, then
$$a(x - x_0) + b(y - y_0) = c - c = 0.$$
- $\gcd(a', b) = \gcd(a, b') = 1$, so
$$a' \mid y - y_0 \text{ and } b' \mid x - x_0.$$
- So $x = x_0 + mb'$ ja $y = y_0 - na'$ holds for some $n, m \in \mathbb{Z}$.
-
$$ax_0 + by_0 = c = ax + by \implies m = n.$$

$\square$

Linear Diophantine equations in two variables

Example

- Solve the Diophantine equation

$$514x + 387y = 2.$$

- First find $\gcd(514, 387)$ by the Euclidean algorithm:

$$\begin{aligned} 514 &= 387 + 127 \\ 387 &= 3 \cdot 127 + 6 \\ 127 &= 21 \cdot 6 + 1 \\ 6 &= 6 \cdot 1 + 0. \end{aligned}$$

- This shows $\gcd(514, 387) = 1 \mid 2$, so the equation has solutions.

Linear Diophantine equations in two variables

Example (Continued)

$$\begin{aligned}
514 &= 387 + 127 \\
387 &= 3 \cdot 127 + 6 \\
127 &= 21 \cdot 6 + 1 \\
6 &= 6 \cdot 1 + 0.
\end{aligned}$$

- Now solve
$$514x + 387y = \gcd(514, 387) = 1$$
by the extended Euclidean algorithm:

$$\begin{aligned}
1 \quad &= 127 - 21 \cdot 6 \\
&= 127 - 21 \cdot (387 - 3 \cdot 127) \quad &&= 64 \cdot 127 - 21 \cdot 387 \\
&= 64 \cdot (514 - 387) - 21 \cdot 387 \quad &&= 64 \cdot 514 - 85 \cdot 387.
\end{aligned}$$

Linear Diophantine equations in two variables

Example (Continued)

- $$1 = 64 \cdot 514 - 85 \cdot 387.$$
- So
$$2 = 2(64 \cdot 514 - 85 \cdot 387) = 128 \cdot 514 - 170 \cdot 387.$$
- Answer: The Diophantine equation
$$514x + 387y = 2$$
has infinitely many solutions,
$$(x, y) = (128, -170) + n(387, -514).$$

Linear Diophantine equations in two variables

Example

- Solve the Diophantine equation

$$112x + 49y = 2.$$

- First find $\gcd(112, 49)$ by the Euclidean algorithm:

$$\begin{aligned} 112 &= 2 \cdot 49 + 14 \\ 49 &= 3 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 + 0. \end{aligned}$$

- This shows $\gcd(112, 49) = 7 \nmid 2$, so the equation has no integer solutions.

Congruency

Definition

- Let n be a positive integer.
- If $n \mid (a - b)$, then we say $a \equiv b \pmod{n}$.
- In words: a and b are **congruent** modulo n.

- Congruence modulo n is an equialence relation on $\mathbb{Z}$.
  - Reflexive: $\forall a \in \mathbb{Z} : n \mid 0 = a - a$.
  - Symmetric: $\forall a, b \in \mathbb{Z}$: If $n \mid a - b$ then $n \mid -(a - b) = b - a$.
  - Transitive:

$$\forall a, b, c \in \mathbb{Z} : \text{ If } n \mid a-b \text{ and } n \mid b-c, \text{ then } n \mid (a-b)+(b-c) = a-c.$$

Congruency and remainders

Fact: $a \equiv b \pmod{n}$ if and only if a and b have the same remainder when divided by n, i.e.

$$(a \bmod n) = (b \bmod n).$$

Example

- $4 \equiv 16 \pmod{12}$; The clock hands are in the same position at 4:00 and 16:00.
- $7654 \equiv 1854 \equiv 54 \pmod{100}$: Same last 2 digits
- $67 \equiv 99 \equiv 1 \pmod{2}$: Odd numbers (remainder 1)
- $29 \equiv 19 \equiv 9 \equiv -1 \equiv -11 \pmod{10}$, all have remainder 9

Congruence class

Definition

- The **congruence class** of $a \in \mathbb{Z}$ modulo n is

$$[a]_n = \{b \in \mathbb{Z} : a \equiv b \pmod{n}\} \subseteq \mathbb{Z}.$$

Example

- $[4]_{10} = \{\dots, -16, -6, 4, 14, 24, \dots\}$
- $[4]_{12} = \{\dots, -20, -8, 4, 16, 28, \dots\}$

Representatives

- All elements of a congruence class are **representatives** of that class.
- Each congruence class has precisely one representative in $\{0, 1, \ldots, n-1\}$. We can call it the *canonical* representative.
- Note: $[n]_n = [0]_n$, and $[-1]_n = [n-1]_n$.

Example

$[27]_{11}$ is a congruence class all right, but its canonical representation is $[5]_{11}$. Note that 27 mod 11 = 5.

Definition

- The **set** of all congruence classes modulo $n \in \mathbb{Z}$ is denoted $\mathbb{Z}_n$ (or $\mathbb{Z}/n\mathbb{Z}$).
- $$\mathbb{Z}_n = \{[0]_n, [1]_n, \cdots, [n-1]_n\}.$$

Addition and multiplication of congruence classes

- For $n \in \mathbb{N} \setminus \{0\}$ and $a, b \in \mathbb{Z}$, define:

$$[a]_n + [b]_n = [a+b]_n$$
$$[a]_n[b]_n = [ab]_n$$

- Note: If $a = pn + r$, $b = qn + s$, then

$$[a+b]_n = [(p+q)n + r + s]_n = [r+s]_n$$
$$[ab]_n = [pnqn + pns + qnr + rs]_n = [rs]_n,$$

  so the sum and product really only depend on the congruence classes of a and b modulo n (these operations are *well-defined*)

- Example: $[4]_3 + [5]_3 = [9]_3 = [3]_3 = [1]_3 + [2]_3$.

Addition and multiplication of congruence classes

Example

- We get addition and multiplication tables as follows in

$$\mathbb{Z}_3 = \{[0]_3, [1]_3, [2]_3\} :$$

$+_3$	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

$\times_3$	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

.

We left out the n subscript from all congruence classes, with the understanding that it is known from the context.

Addition and multiplication of congruence classes

Theorem

The following laws hold for $a, b, c \in \mathbb{Z}_n$:

- $a + b = b + a$ *and* $ab = ba$ (commutativity)
- $a + (b + c) = (a + b) + c$ *and* $a(bc) = (ab)c$ (associativity)
- $a + [0] = a$ *and* $a \cdot [1] = a$ (neutral elements)
- *For each a there exists $-a$ s.t. $a + (-a) = [0]$.* (additive inverse)
- $a(b + c) = ab + ac$ (distributivity)

- Note: $a, b, [0], [1]$ are *congruence classes*; not integers.
- These are the axioms of a **commutative ring with a unit**.
  - In some sources, this is called a **commutative ring**, or even just a **ring**.
- The set $\mathbb{Z}_n$ is called **the ring of integers modulo n**.

Differences between $\mathbb{Z}$ and $\mathbb{Z}_n$

- The table did not talk about *multiplicative* inverses.
- b is a **multiplicative inverse of a** if $ab = ba = 1$. In this case we say that a is **invertible**
- In $\mathbb{Z}$, only ± 1 have multiplicative inverses.
- In $\mathbb{Z}_n$, other elements **can** have inverses too. Perhaps some elements have, and other do not!
- Example: $[2]_5 \cdot [3]_5 = [1]_5$, so $[2]_5$ and $[3]_5$ are inverses in $\mathbb{Z}_5$.

Examples: $\mathbb{Z}_n$ multiplication tables, n prime

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Observations (**other** than the zero row):

- Every row contains a 1, so every element has an inverse
- Every row contains only one 1
- Every row contains $0, \ldots, n-1$ permuted

Example: $\mathbb{Z}_n$ multiplication table, n composite

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Observations:

- Some rows contain a 1, e.g. $[5] \cdot [5] = [1]$. Element $[5]$ is invertible
- Some rows don't contain 1, but contain some zeros: $[3] \cdot [4] = [0]$
- $[3]$ and $[4]$ are *divisors of zero*, and not invertible

Differences between $\mathbb{Z}$ and $\mathbb{Z}_n$

- A commutative ring with a unit, where all non-zero elements have an inverse, is called a **field**.
- Example: $\mathbb{R}$ and $\mathbb{Q}$ are fields.

Theorem

- *Let p be a prime.*
- *Then $\mathbb{Z}_p$ is a field.*

Proof.

- Let $0 < a < p$, so $[a]_p \neq [0]_p$. Then $\gcd(p, a) = 1$.
- By Bezout's identity, $xp + ya = 1$ has an integer solution.
- Then $ya \equiv 1 \pmod{p}$, so $[y]_p$ is an inverse of $[a]_p$. $\square$

Differences between $\mathbb{Z}$ and $\mathbb{Z}_n$

- In $\mathbb{Z}_n$ it is not true that $ab = ac \Rightarrow b = c$.
- In fact, this is true if and only if a is invertible.
- $[x]$ is invertible in $\mathbb{Z}_n$ if and only if $\gcd(x, n) = 1$.

Example

- In $\mathbb{Z}_6$, $[2] \cdot [4] = [2] \cdot [1]$, but $[4] \neq [1]$.

Congruence equations

- When does $b \equiv ax \pmod{n}$ have a solution?
- If $\gcd(a, n) \neq 1$, then we must have $\gcd(a, n) \mid b$.
- In such case, divide the equation by $\gcd(a, n)$.

Theorem

- *Assume* $\gcd(a, n) = 1$.
- *Then* $ax \equiv b \pmod{n}$ *has a unique solution (modulo n).*

Proof.

- $[a]$ has an inverse $[a]^{-1}$ in $\mathbb{Z}_n$.
- $[a][x] = [b] \Rightarrow [x] = [a]^{-1}[a][x] = [a]^{-1}[b]$. $\square$

Congruence equations

Example

- The invertible elements in $\mathbb{Z}_{10}$ are $[1], [3], [7], [9]$.
- Their inverses are

$$[1]^{-1} = [1],\ [3]^{-1} = [7],\ [7]^{-1} = [3],\ [9]^{-1} = [9]$$

respectively. Notice: $[9] = -[1]$.

Congruence equations

Example

- The invertible elements in $\mathbb{Z}_{12}$ are $[1], [5], [7], [11]$.
- They are all their own inverses.
- We can solve the congruence

$$7x \equiv 9 \pmod{12}$$

  by **multiplying with the inverse** of 7, modulo 12.
-
$$x \equiv 7 \cdot 7x \equiv 7 \cdot 9 \equiv 63 \equiv 3 \pmod{12}.$$

Handy rule: Divisibility by three

In the positional notation, in base 10, the number "*abcdef*" means

$$x = 10^5 \cdot a + 10^4 \cdot b + 10^3 \cdot c + 10^2 \cdot d + 10 \cdot e + f.$$

Claim: $x \equiv a + b + c + d + e + f \pmod 3$.

Proof: Because $10 \equiv 1 \pmod 3$, also $10^k \equiv 1^k \equiv 1$. Thus

$$x \equiv 1 \cdot a + 1 \cdot b + 1 \cdot c + 1 \cdot d + 1 \cdot e + 1 \cdot f.$$

Corollary: $3 \mid x$ iff 3 divides the sum of digits in x

Example: $452123 \equiv 4 + 5 + 2 + 1 + 2 + 3 \equiv 17 \equiv 2 \pmod 3$.

Similar rule for divisibility by 9. But not other numbers, in base 10. Think why.

Exponents modulo n

Example

- What is the remainder of 3^{13} when divided by 100?
- Division algorithm: $3^{13} = 100q + r$, so $[r]_{100} = [3^{13}]_{100}$.
- We save time by not computing 13 multiplications, but doing *repeated squaring* in $\mathbb{Z}_{100}$:

$$
\begin{aligned}
[3]^2 &= [9] \\
[3]^4 &= [9]^2 = [81] \\
[3]^8 &= [81]^2 = [6561] = [61] \\
[3]^{13} &= [3]^8 \cdot [3]^4 \cdot [3]^1 = [61][81][3] = [14823] = [23].
\end{aligned}
$$

- So the remainder is 23.

Exponents modulo n

- If the exponent is very large, then even repeated squaring is inconvenient.

Example

- Can we compute $[3]_{13}^{100}$?
- Yes, because we are lucky! $[3]^3 = [27] = [1]$.

$$[3]^{100} = ([3]^3)^{33} \cdot [3] = [1]^{33} \cdot [3] = [3]$$

- So the remainder is 3.

- It would help if we had a *systematic* way to find a number k such that

$$a^k \equiv 1 \pmod{n}.$$

(if $\gcd(a, n) = 1$).

Fermat's little theorem

Theorem

Let p be a prime and $a \not\equiv 0 \pmod{p}$. Then $a^{p-1} \equiv 1 \pmod{p}$.

Proof.

- Each $[a][x] = [b]$ has a unique solution if $[b] \neq [0]$.
- So
$$\{[1], [2], \ldots [p-1]\} = \{[a][1], [a][2], \ldots [a][p-1]\}.$$
- Thus
$$[(p-1)!] = \prod_{i=1}^{p-1} [i] = \prod_{i=1}^{p-1} [a][i] = [a]^{p-1}[(p-1)!].$$
- But $p \not| (p-1)!$, so $(p-1)!$ is invertible modulo p.
- It follows that $[1]_p = [a]_p^{p-1}$. $\square$

Fermat's little theorem

Example

We check Fermat's little theorem in $\mathbb{Z}_7$:

- $1^6 = 1$
- $2^6 = (2^3)^2 = 1^2 = 1$
- $3^6 = (3^3)^2 = (-1)^2 = 1$
- $4^6 = (-3)^6 = 3^6 = 1$
- $5^6 = (-2)^6 = 2^6 = 1$
- $6^6 = (-1)^6 = 1^6 = 1$

Euler's theorem

- How do we compute powers modulo a non-prime n?
- The proof of Fermat's little theorem suggests a generalization.

Definition

- Let $n \in \mathbb{N}$.
- The Euler function $\varphi(n)$ is the number of elements

$$0 \leq i < n \text{ such that } \gcd(n, i) = 1.$$

- Note: $\varphi(n) = n - 1$ if and only if n is prime.
- Equivalently, $\varphi(n)$ is the number of invertible elements in $\mathbb{Z}_n$.

Euler's theorem

Theorem

- *Let* $n \in \mathbb{N}$, *and* $\gcd(a, n) = 1$.
- *Then* $a^{\varphi(n)} \equiv 1 \pmod{n}$.

- The proof closely follows that of Fermat's little theorem.
- It follows that, if $b = q\varphi(n) + r$, then $a^b \equiv a^r \pmod{n}$.

Euler's φ function

- If $n = p^k$ is a power of a prime, then

$$\begin{aligned}\varphi(n) &= \mid \{0 \leq i < n : \gcd(n, i) = 1\}\mid \\ &= p^k - \{pj : 0 \leq j < p^{k-1}\}\mid \\ &= (p-1)p^{k-1}.\end{aligned}$$

- If $\gcd(a, b) = 1$, then $\varphi(ab) = \varphi(a)\varphi(b)$. (Proof omitted.)
- Thus,

$$\varphi(p_1^{k_1} \cdots p_r^{k_r}) = (p_1 - 1) \cdots (p_r - r) \cdot p_1^{k_1 - 1} \cdots p_r^{k_r - 1}$$

- If we can factorize n, then we can also compute powers modulo n more efficiently than before.

Euler's φ function

Example

- How many integers in $[0, 10200]$ are relatively prime to 10200?
- First factorize

$$\begin{aligned} 10200 &= 2 \cdot 5100 &&= 2^2 \cdot 2550 &&= 2^3 \cdot 1275 \\ &= 2^3 \cdot 3 \cdot 425 &&= 2^3 \cdot 3 \cdot 5 \cdot 85 &&= 2^3 \cdot 3 \cdot 5^2 \cdot 17. \end{aligned}$$

- Thus we get

$$\begin{aligned} \varphi(10200) &= (2-1)2^2 \cdot (3-1) \cdot (5-1)5 \cdot (17-1) \\ &= 2^{2+1+2+4} \cdot 5 \\ &= 512 \cdot 5 = 2560. \end{aligned}$$

Euler's φ function

Example (Continued)

- $$\varphi(10200) = 2560.$$
- By Euler's theorem,
  $$a^{2560} \equiv 1 \pmod{10200}$$
  for all a with $\gcd(10200, a) = 1$.

- If $m \equiv 1 \pmod{\varphi(n)}$ and $\gcd(a, n) = 1$, then $a^m \equiv a \pmod{n}$.

RSA cryptography

- In 1978, Ron Rivest, Adi Shamir and Leonard Adleman demonstrated the RSA cryptography scheme.
- It allows anybody with a *public* key to send messages to Alice.
- Alice has a *private* key, with which she can read the secret message.
- RSA cryptograpy is considered secure in practice.
- Breaking the crypto (i.e. reading the message without the private key) is equally difficult as computing $\varphi(n)$ for a large number n.

RSA cryptography

- Anybody with a *public* key (k, n), can transmit a message $s \in \mathbb{Z}_n$ to Alice, by sending the message $s^k \in \mathbb{Z}_n$. This is easy to compute.
- Alice can compute
$$s = s^{k\ell} = (s^k)^\ell,$$
if $k\ell \equiv 1 \pmod{\varphi(n)}$.
- ℓ is the inverse of k modulo $\varphi(n)$, **and Alice knows $\varphi(n)$.**
- Breaking the crypto (i.e. reading the message without the private key) is equally difficult as computing $\varphi(n)$ for a large number n.

RSA cryptography

- Breaking the RSA crypto is equally difficult as computing $\varphi(n)$ for a large number n.
- This is equivalent to prime factorizing n
- No efficient algorithm is known for this *on a classical computer*.
- Sage demo: https://sagecell.sagemath.org/?q=iyqbfg
- Peter Shor showed in 1993, that primes can in principle be efficiently factorized on a *quantum computer*.
- *If* quantum computers actually start working on a big scale, RSA will be outdated.
- To date, Shor's algorithm has managed to factorize $21 = 7 \times 3$.

RSA cryptography

- Alice generates two large primes p and q secretly.
- She computes $n = pq$ (public knowledge) and $\varphi(n) = (p-1)(q-1)$.
- Alice chooses a number k (public) with $\gcd(k, \varphi(n)) = 1$, and in secret computes its inverse d in $\mathbb{Z}_{\varphi(n)}$.
- Public key: (k, n).
- Alice trusts that the number d remains secret.
  - Computing d from the public key would require first computing $\varphi(n)$, *i.e.* factorizing the large number n.

RSA cryptography

- Mathematical essence: $(s^k)^d = s^{kd} = s^{r\varphi(n)+1} = s$.
  - This is a consequence of Euler's theorem.
- Computational essence 1: It is **easy** to compute s^k from s.
- Computational essence 2: It is **easy** to compute $s = (s^k)^d$ from s^k if you know d.
- Computational essence 3: It is **difficult** to compute s from s^k if you do not know d.

RSA cryptography

- A user Bob who wants to send a message to Alice, first writes that message using the "alphabet" $[0], [1], [2], \ldots, [n-1]$.
- In our example, Bob uses the translation $A = 1, B = 2, C = 3, \ldots$.
  - If n is really large, he can translate more efficiently by encoding more than one letter per symbol, like $AA = 1, AB = 2, \ldots$.
  - To avoid "frequency attacks", Bob might encode common strings into a single symbol.
- Encoding: If Bob wants to communicate the symbol $s \in \mathbb{Z}_n$ to Alice, he instead sends the symbol $s^k \in \mathbb{Z}_n$.

RSA cryptography

- Encoding: If Bob wants to communicate the symbol $s \in \mathbb{Z}_n$ to Alice, he instead sends the symbol $s^k \in \mathbb{Z}_n$.
- Decoding: If Alice receives the symbol $t \in \mathbb{Z}_n$, she knows that the sent symbol was

$$t^d = (s^k)^d = s^{kd} = s^{r\varphi(n)+1} = s.$$

- Cracking the crypto: If we can factorize n, then we can compute $\varphi(n)$, and then compute d from k by solving the diophantine equation

$$1 = kd + \varphi(n)y.$$

Spying example

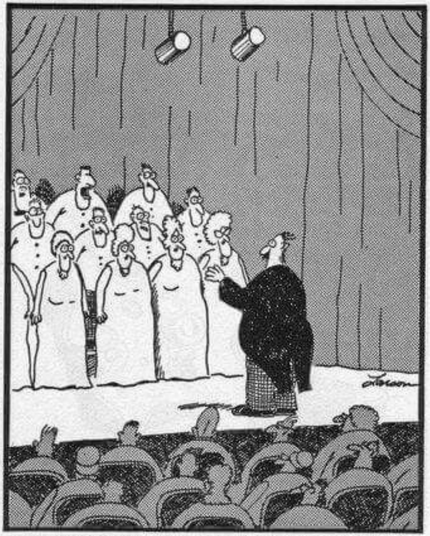
.

- Public key: $(5, 2021)$.
- (We pretend that it were difficult to factor $2021 = 43 \cdot 47$).
- Secret message: `"The cats' names are`

  `1698 1500 1954 1450 1104 1671 0757 0001 1954 0440`

  `and`

  `0432 1104 1450 1681 0249 0440."`



In that one split second, when the choir's last note had ended, but before the audience could respond, Vinnie Conswego belches the phrase, "That's all, folks."