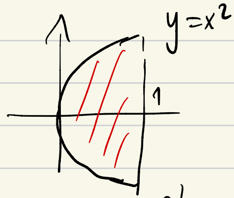
Ändra R till



$$\iint_R \frac{1}{x^2}\,dA = \int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{x^2}\,dy = \int_0^1 2\sqrt{x}\,\frac{1}{x^2}\,dx = \int_0^1 2x^{-3/2}\,dx =$$

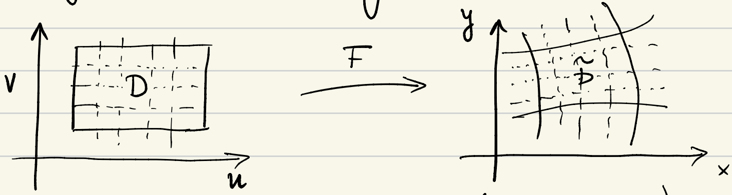
$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 2x^{-3/2}\,dx = \lim_{\varepsilon \to 0^+} \left[\frac{2x^{-1/2}}{-1/2} \right]_\varepsilon^1 =$$

$$= \lim_{\varepsilon \to 0^+} \frac{4}{\sqrt{\varepsilon}} - 4 = \infty$$

Integrabilitet beror på $f(x,y)$ och R.

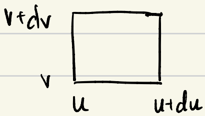
Variabelbyten i multipelintegraler

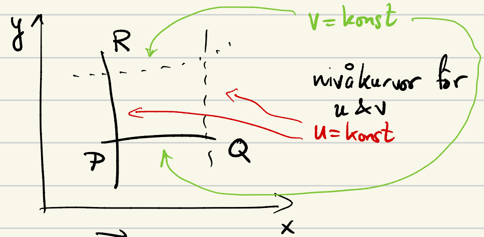
En avbildning $F: \underset{\mathbb{R}^n}{U} \to \underset{\mathbb{R}^n}{W}$ kallas ett variablebyte om den är bijektiv (av rätt typ). Först $\mathbb{R}^2$.



$$F(u,v) = (x(u,v), y(u,v))$$

Om vi vill beräkna $\iint_{\tilde{D}} f(x,y)\,dx\,dy$ med hjälp av variabelbytet $F(u,v)$ behöver vi förstå hur F förändrar areaskalan.





$\vec{PQ} = (dx)\vec{e_1} + (dy)\vec{e_2}$ $\qquad$ $\vec{PR}$ på samma sätt

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \quad ; \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

På PQ är v konstant. Därför $dv = 0$ där

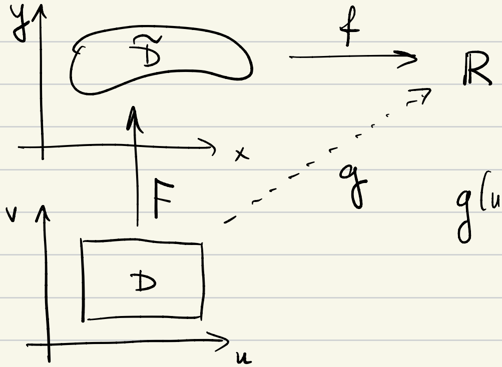
$$\Longrightarrow \vec{PQ} = \left(\frac{\partial x}{\partial u}du\right)\vec{e_1} + \left(\frac{\partial y}{\partial u}du\right)\vec{e_2}$$

Dessutom $\vec{PR} = \left(\frac{\partial x}{\partial v}dv\right)\vec{e_1} + \left(\frac{\partial y}{\partial v}dv\right)\vec{e_2}$

$$dx\,dy \approx \left| \det \begin{pmatrix} \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv \end{pmatrix} \right| =$$

$$= \underbrace{\left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right|}_{\text{Jacobianen } \frac{\partial(x,y)}{\partial(u,v)}} du\,dv = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Alltså

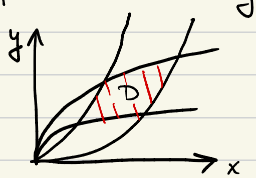


$$g(u,v) = f(F(u,v)) = f(x(u,v), y(u,v))$$

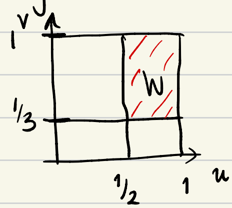
$$\iint_{\tilde{D}} f(x,y)\,dx\,dy = \iint_D g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Ex Beräkna arean av området som begränsas av de fyra parablerna $y = x^2$, $y = 2x^2$, $x = y^2$ och $x = 3y^2$.

Lösning:



$u = x^2/y \qquad v = \dfrac{y^2}{x}$



Lägg märke till att $\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| = 1 \Big/ \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

$\partial u/\partial x = \dfrac{2x}{y}$; $\partial u/\partial y = -x^2/y^2$; $\dfrac{\partial v}{\partial x} = -y^2/x^2$ och $\dfrac{\partial v}{\partial y} = \dfrac{2y}{x}$

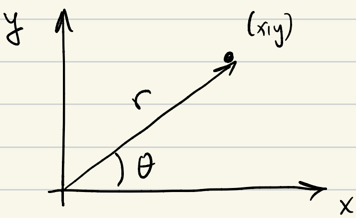
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -y^2/x^2 \\ -x^2/y^2 & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{3}$$

$$\text{Arean} = \iint_D 1 \, dxdy = \iint_W \frac{1}{3} \, dudv = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{9} \text{ a.e}$$

<u>Mycket viktigt variabelbyte</u>

Polära koordinater



$r > 0$

$0 \leq \theta < 2\pi$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r^2 = x^2 + y^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\boxed{dxdy = r\,drd\theta}$$

Alltså, $\iint_D f(x,y)\,dxdy = \iint_{\tilde{D}} g(r,\theta)\, r\, drd\theta$

Ex Låt $D = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 < 4\}$

Beräkna $I = \iint_D \frac{1}{x^2+y^2}\,dx\,dy$.

Lösning: I polära koordinater är

$$I = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\, r\,dr\,d\theta = 2\pi \int_1^2 \frac{1}{r}\,dr =$$

$$= 2\pi \left[\ln r\right]_1^2 = 2\pi \ln 2.$$

Variabelbyten funkar på samma sätt i högre dimensioner

T.ex. $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$ $\quad dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$

Ex: Beräkna volymen hos ellipsoiden E

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \qquad (a,b,c > 0)$$

Lösning: $u = \frac{x}{a}$; $v = \frac{y}{b}$; $w = \frac{z}{c}$

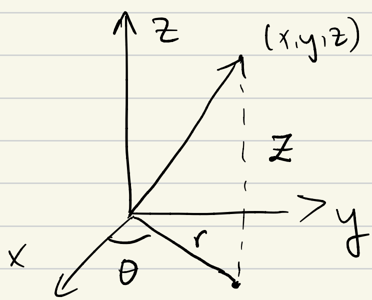
$\Longrightarrow u^2 + v^2 + w^2 \leq 1$ En sfär S med radie 1.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\iiint_E 1\,dx\,dy\,dz = \iiint_S abc\,du\,dv\,dw = abc \cdot (\text{volymen hos } S)$$

$$= \frac{4\pi abc}{3}$$

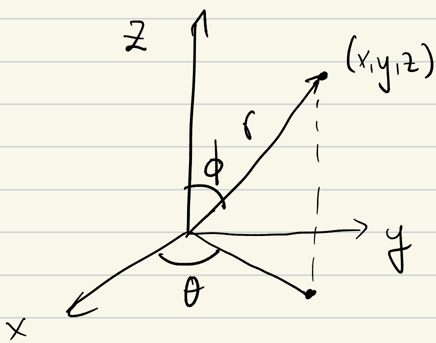
Cylindriska koordinater



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} 0 < r \\ 0 \le \theta < 2\pi \\ z \in \mathbb{R} \end{array}$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Sfäriska koordinater



$$0 < r$$
$$0 \le \theta < 2\pi$$
$$0 < \phi < \pi$$

$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

Om man beräknar Jacobianen så får man

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$$

<u>Ex</u> Beräkna volymen hos en sfär med radie R.

$$S_R = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq R^2 \}$$

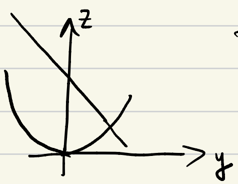
<u>Lösning:</u>

$$\iiint_{S_R} 1 \, dx\,dy\,dz = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\phi \, d\phi \, d\theta \, dr =$$

$$= \int_0^R \int_0^{2\pi} \left[-r^2 \cos\phi \right]_0^{\pi} d\theta \, dr = \int_0^R \int_0^{2\pi} 2r^2 \, d\theta \, dr =$$

$$= 4\pi \int_0^R r^2 \, dr = 4\pi \left[\frac{r^3}{3} \right]_0^R = \frac{4\pi R^3}{3} \text{ v.e}$$

<u>Ex</u>: Beräkna volymen av den kropp som begränsas av planet $z = 3-2y$ och paraboloiden $z = x^2+y^2$



Planet och paraboloiden skär då

$$x^2+y^2 = 3-2y$$

$$x^2+y^2+2y-3 = 0$$

$$x^2+(y+1)^2-4 = 0 \iff x^2+(y+1)^2 = 4$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+(y+1)^2 \leq 4 \}$$

$$\text{Volym} = \iint_D 3-2y-(x^2+y^2) \, dx\,dy = \iint_D 4-x^2-(y+1)^2 \, dx\,dy =$$

$$= \begin{bmatrix} x = r\cos\theta & ; & 0 \leq r < 2 \\ y+1 = r\sin\theta & & 0 \leq \theta < 2\pi \end{bmatrix} = \int_0^{2\pi} \int_0^2 (4-r^2) r \, dr \, d\theta =$$

$$= \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^4}{4} \right]_0^2 d\theta = 2\pi (8-4) = 8\pi \text{ v.e}$$