1A Introductory exercises

This first set contains problems to be solved during the first exercise session (1A). You obtain exercise points for the session by attending the session.

Sets

Recall the notion of a *set* as an unordered collection of distinct elements. Check lecture notes pages 3—10 as needed. Recall that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the sets of natural numbers (that is, non-negative integers), integers, rational numbers, and real numbers. Note that our natural numbers begin from zero (some people use a different definition where they begin from one).

The *set builder notation* (page 4) is a method for specifying a set without listing its elements explicitly. For example,

$$\{x \in \mathbb{Z} \ : \ 4 \leq x \leq 7\}$$

means (by definition) the set of those integers whose value is at least 4 but no more than 7, that is,

$$\{4, 5, 6, 7\}.$$

1A1 (Intervals of integers) How many elements do the following sets contain? Here a and b are some arbitrary integers (possibly $a < b$ but possibly not). Try to make the resulting expression as simple as possible.

(a) $\{x \in \mathbb{Z} \ : \ 10 \leq x \leq 15\}$ (Hint: You can start by listing the elements.)

(b) $\{x \in \mathbb{Z} \ : \ 10 < x < 15\}$

(c) $\{x \in \mathbb{Z} \ : \ 10 \leq x < 15\}$

(d) $\{x \in \mathbb{Z} \ : \ 20 \leq x \leq 15\}$

(e) $\{x \in \mathbb{Z} \ : \ a \leq x \leq b\}$

(f) $\{x \in \mathbb{Z} \ : \ a < x < b\}$

(g) $\{x \in \mathbb{Z} \ : \ a \leq x < b\}$

1A2 (Ellipsis and counting) The *ellipsis notation* is another handy way of writing a large set, but it is also dangerous. The meaning of the three dots is supposed to be “continuing in the same manner”, and it is up to the reader to understand what is “the same manner”. Easy examples are some consecutive integers like $\{5, 6, 7, \ldots, 12\}$ and some consecutive even integers like $\{2, 4, 6, 8, \ldots, 20\}$.

Count the elements of the following sets. List their elements if convenient. “Counting” means “determining the number of something” (not necessarily by listing one by one).

(a) $\{10, 12, 14, \ldots, 20\}$

(b) $\{0, 2, 4, \ldots, 100\}$

(c) $\{1, 4, 9, \ldots, 100\}$ (meaning: squares of consecutive integers)

(d) $\{1, 4, 9, \ldots, 1\,000\,000\}$

(e) $\{1, 4, 9, \ldots, a\}$, where a is a square of an integer.

1A3 (Operations of sets) Recall the notions of closed, open and half-open intervals (of real numbers).

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} \;:\; a \le x \le b\} \\
]a, b[&= \{x \in \mathbb{R} \;:\; a < x < b\} \\
[a, b[&= \{x \in \mathbb{R} \;:\; a \le x < b\} \\
]a, b] &= \{x \in \mathbb{R} \;:\; a < x \le b\}
\end{aligned}$$

Caveat: Some people use the notation (a, b) for the open interval (this could be confused with the similar notation for an ordered pair of two numbers).

In the following, a, b, c, d are real numbers.

(a) What is $[a, a]$? How many elements does it contain?

(b) What is $]a, a[$? How many elements does it contain?

(c) What is $[a, b]$ if $b < a$?

(d) What is $[5, 10] \cup [7, 12]$? Can it be expressed as an interval?

(e) What is $[5, 10] \cap [7, 12]$? Can it be expressed as an interval?

(f) What is $[5, 10] \cup [15, 20]$? Can it be expressed as an interval?

(g) What is $[5, 10] \cap [15, 20]$? Can it be expressed as an interval?

(h) When exactly is $[a, b] \cup [c, d]$ an interval?

(i) When exactly is $[a, b] \cap [c, d]$ an interval?

(j) What is $[0, 10] \setminus [4, 6]$? (Note the set difference operation.) Express it as an union of two intervals, be careful with the endpoints.

1A4 (Set membership and logic) A *statement* is a claim that is either true or false. If it contains variables (like x), its truth value may depend on the value of x: for example, $x < 3$ may be true or false depending on what x is. Typical mathematical statements are equality ($x = 4$), inequalities ($x < 4$), and set membership ($x \in \{2, 4, 6\}$).

Statements can also be combined into bigger statements by *connectives*. The simplest connectives are $\wedge$ (**and**, meaning that both components are true) and $\vee$ (**or**, meaning that at least one of the components is true — possibly both). To aid your memory, note that $\wedge$ looks a little like an A ("and").[1]

Now we explore the connection between connectives and set membership. Let[2]

$$\begin{aligned} A &= \{1, 2, 3, \ldots, 10\} \\ B &= \{x \in A \ : \ x \leq 5\} \\ E &= \{x \in A \ : \ x \text{ is even}\} \end{aligned}$$

(a) Write out the sets B and E explicitly or using the ellipsis notation.

(b) Write out the intersection $B \cap E$.

(c) Write out the union $B \cup E$.

(d) Consider the "and" (conjunction) statement

$$(x \leq 5) \ \wedge \ (x \text{ is even}).$$

For which of the values $x \in A$ is this statement true? (Hint: Consider individual values of x. For example, if $x = 1$, is the statement true of false? Then what if $x = 2$?) Is this equal to one of sets seen in (b) and (c)?

(e) Consider the "or" (disjunction) statement

$$(x \leq 5) \ \vee \ (x \text{ is even}).$$

For which of the values $x \in A$ is this statement true? Is this equal to one of sets seen in (b) and (c)?

(f) Do you notice some useful connection between the symbols $\cap, \cup, \wedge, \vee$?

1A5 (Boolean searches) We have a database of a million people. We can think of this as a set A. For each person, the database contains information such as citizenship, which could be Finnish, Swedish, or something else (each person has only one citizenship). The user can form a query like "x is Finnish", and the database system then goes through every person in the database, checks if this statement is true for that person, and returns the list of matching persons. A query can also combine statements using connectives **and** and **or**. The database contains 200 000 Finnish citizens and 100 000 Swedish citizens.

A novice user wants to count all Finns **and** Swedes, and writes the query: "x is Finnish **and** x is Swedish". To the user's utter surprise, the result set is empty (zero persons). Explain.

[1] Those who excel in Latin may want to remember that *vel* means "or", and $\vee$ looks like a V. Finns might recall that *vai* is a bit like *tai*.

[2] An integer is *even* if its value is two times some integer, e.g. 8 is even because $8 = 2 \times 4$. Otherwise it is *odd*. For example, $0, 2, 4, 6$ are even and $1, 3, 5, 7$ are odd.

1A6 (Subsets) If A and B are two sets, then A is a *subset* of B (written $A \subseteq B$) if every element of A is also an element of B. For example $\{1, 3\} \subseteq \{1, 2, 3, 4\}$. Or, in other words: A does *not* contain any element that would *not* also be in B. Note that A could be empty (indeed, an empty set does not contain any elements that are not in B), and also A and B could be the same set.

(a) List all subsets of $A = \{1, 2\}$. (There should be four of them.)

(b) List all subsets of $B = \{1, 2, 3\}$ using the following method: first list the zero-element subsets, then the one-element subsets, then the two-element subsets, and then the three-element subsets. (Remember that order does not matter in a set: $\{b, a\}$ is the same set as $\{a, b\}$.) How many of each size do you obtain? How many in total?

(c) List all subsets of $B = \{1, 2, 3\}$ using the following method: Consider, in turn, each of the sets you listed in (a). For each one, either add the element 3 or not. How many different sets do you obtain and why? Does this method necessarily produce all the subsets, and each one exactly once?

(d) List and count all subsets of $C = \{1, 2, 3, 4\}$ using a method similar to (b). Try to be systematic. How do you make sure that you list every possibility, but each one only once?

(e) If you listed all subsets of $C = \{1, 2, 3, 4\}$ using a method similar to (c), that is, taking all the subsets of $\{1, 2, 3\}$ and then either adding 4 or not, how many would there be? (You can actually do the listing if you want, but it is not necessary.)

(f) What happens to the number of subsets when the original set is grown by one element?

(g) How many different subsets does an n-element set have? Try to write a concise expression. How many different subsets does a 10-element set have?

(h) Try your expression from the previous item for the subsets of a one-element set $\{1\}$, and for a zero-element set.