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2B Proof techniques

2B1 (Cartesian products) For each of the following equations, either prove it true, or prove it false by a concrete counterexample (an element of one side of the equation that is not an element of the other side).

(a) $(\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$

(b) $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) = \mathbb{R} \times \mathbb{R}$

2B2 (Parity) Prove that if $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even. Hint: Direct proof by cases: case 1 for n even, and case 2 for n odd.

2B3 (Multi-way De Morgan) On the lectures we learned about De Morgan's law for two propositions. An obvious-looking generalization is the following claim (where $p_1, \ldots, p_n$ are arbitrary propositions):

$$\big(\neg(p_1 \vee \ldots \vee p_n)\big) \iff \big((\neg p_1) \wedge \ldots \wedge (\neg p_n)\big).$$

(a) Prove the claim for $n = 3$ directly by a truth table of 8 rows.

(b) For $n = 10$, how many rows would you need, if you wrote the full truth table explicitly? Can you characterize in just a few words how the table would look (on which rows would the results of the LHS and the RHS be "true", and on which rows would they be "false")?

(c) Prove the claim for all integers $n \geq 2$ by induction. (Hint: A conjunction of n propositions can be decomposed into a two-way conjunction of one proposition and the conjunction of the remaining $n - 1$.)

2B4 (Sums of powers)

(a) Prove by induction that $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$ for all integers $n \geq 0$.

(b) Prove by induction that $\sum_{j=0}^{n} 3^j = (3^{n+1} - 1)/2$ for all integers $n \geq 0$.

(c) Prove by induction that $\sum_{j=0}^{n} 10^j = (10^{n+1} - 1)/9$ for all integers $n \geq 0$. Calculate a few first values of the LHS and RHS. Does the result look obvious?

2B5 (Faulty induction) Consider all one-colored socks in the world. (For brevity we will just call them "socks". We do not consider socks that contain multiple colors.) We assume here that color is a well-defined property: the colors of any two socks are either same or different. Here is a purported proof that *all socks in the world have the same color.*

"Proof." For every integer $n \geq 1$, let $P(n)$ be the claim "Every collection of n socks is unicolored" (i.e. the socks in the collection have the same color). We prove by induction that $P(n)$ is true for all n. In particular, $P(n)$ is true when n is the number of all socks in the world. Find the error in this proof.

- Base case: Clearly $P(1)$ is true, because in any one-sock collection there is only one color.
- Induction step: Suppose that for some n, $P(n)$ is true. We will prove that $P(n+1)$ is then also true. Consider any collection of $n+1$ socks, and name its socks $s_1, s_2, \ldots, s_{n+1}$. By the induction hypothesis, the first n socks $(s_1, \ldots, s_n)$ all have the same color. Also by the induction hypothesis, the last n socks $(s_2, \ldots, s_{n+1})$ have the same color. Because s_1 has the same color as s_2, and all the remaining socks also have the same color as s_2, clearly all $n+1$ socks have the same color.
- By the induction principle, $P(n)$ is true for all n. The proof is complete.

2B6 (Fibonacci parity) The Fibonacci numbers are the sequence $f_0, f_1, f_2, \ldots$ where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ when $n \geq 2$.

(a) Calculate the first ten Fibonacci numbers $f_0, \ldots, f_9$ and circle those that are even.

(b) Study the indices n of your circled numbers, and guess a very simple rule that allows you, by looking at the index n to decide whether f_n is even or odd (without having to calculate f_n).

(c) Prove your rule for all $n \in \mathbb{N}$ by induction.

Again the "challenge" problem is worth an extra point. That is, by doing any six problems (whether it includes the challenge problem or not) you gain six points, which is considered 100% of this set. By doing all seven you gain seven points, which is considered $7/6 = 116\frac{2}{3}\%$.

2B7 (** CHALLENGE: Golomb rulers) A **Golomb ruler** is a set $A \subseteq \mathbb{Z}$, where the smallest element is zero, and each pair of elements has a different distance (between its two elements).[1] Distance means arithmetic difference. For example, $\{0, 1, 2\}$ is *not* a Golomb ruler, because between 0 and 1 there is the same distance as between 1 and 2. But $\{0, 1, 5\}$ *is* a Golomb ruler, because all its pairs have different distances: $1-0=1$, $5-1=4$ and $5-0=5$. (We do not consider distances of elements to themselves — they would of course all be zero — and we only consider the positive

[1] The usual definition does not require the smallest element to be zero, but here we do so for simplicity.

distances $b - a$ where $b > a$.) The *length* of a ruler is the distance between its smallest and largest element. Golomb rulers have real-world applications in error-correcting codes, X-ray crystallography, radio frequency selection and radio antenna placement. One common problem is to find the *shortest* Golomb ruler of a given cardinality. The elements are called “marks” in analogy to actual rulers.

(a) Find a Golomb ruler with three marks and length 3.

(b) Prove that there is no Golomb ruler with three marks and length smaller than 3.

(c) Find a shortest possible Golomb ruler with four marks. Prove that there is no shorter one. (Hint: It is probably a good idea to break into cases. There are many different ways of doing that.)