3A Relations

3A1 (Parity arithmetic) The *parity* of an integer means whether it is even or odd. We say that an integer n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$, and *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

For simplicity, we assume now (without actually proving it) that every integer is either even or odd, but not both. (We may return to this topic later on the course.)

Find the parities of $a+b$ and ab in each of these cases, when a and b are arbitrary integers. You can start by calculating some examples (if you need), but you should then prove that your claims hold for *all* integers. (Hint: Induction is probably not needed. Clever use of quantifiers should be enough.)

(a) a is even and b is even

(b) a is even and b is odd

(c) a is odd and b is even

(d) a is odd and b is odd

Is it possible that $a + b$ and ab are both odd? Is it possible that both are even?

3A2 (Combination of equivalences)

(a) Prove or disprove: if R and S are equivalence relations, then $R \wedge S$ is an equivalence relation.

(b) Prove or disprove: if R and S are equivalence relations, then $R \vee S$ is an equivalence relation.

3A3 (Bit strings) An n-fold Cartesian product of a set S by itself can be denoted as

$$S^n = S \times S \times \ldots \times S.$$

In particular, $\{0, 1\}^n$ is the set of all strings (tuples) of n bits. (A *bit* is one of the two integers 0 and 1.) We define a relation R in $\{0, 1\}^n$ such that $R(x, y)$ holds if and only if the strings x and y contain the same number of ones.

(a) Prove that R is an equivalence.

(b) Let $\mathbf{0} = 00\ldots0$ be the string of n zeros. What is the equivalence class $[\mathbf{0}]$?

(c) How many equivalence classes does R have?

3A4 (Divisibility) When a and b are positive integers, we say that a *divides* b (written $a \mid b$) if there is an integer k such that $b = ka$.

Prove that $\mid$ is an order, but not a total order, on positive integers.

You may notice that our definition really talks about multiplication, not about division. We'll talk more about this later on the course, in number theory. Instead of "divides", you could read $a \mid b$ as "b is a multiple of a" (or more precisely, "an integer multiple"), or "a is a factor of b".

3A5 (Same set, different orders) We define two different orders on $\mathbb{Z}^2$, the *pointwise order*

$$(a_1, a_2) \preceq_P (b_1, b_2) \quad \text{if and only if} \quad a_1 \le b_1 \ \wedge \ a_2 \le b_2$$

and the *lexical order*

$$(a_1, a_2) \preceq_L (b_1, b_2) \quad \text{if and only if} \quad (a_1 < b_1) \vee (a_1 = b_1 \ \wedge \ a_2 \le b_2).$$

(a) Prove that both are orders.

(b) Which one(s) of them are total orders? Prove it.

(c) Prove that for all points $a, b \in \mathbb{Z}^2$, $a \preceq_P b$ implies $a \preceq_L b$.

(d) Visualize both orders by taking some point, say $b = (4, 2)$, and drawing the sets $\{a \in \mathbb{Z}^2 \ : \ a \preceq b\}$, where $\preceq$ is either $\preceq_P$ or $\preceq_L$.

(e) Give an example of a third order on $\mathbb{Z}^2$ (different from both $\preceq_P$ and $\preceq_L$) and visualize it.

3A6 (Manhattan) We define the *Manhattan distance* between points in $\mathbb{Z}^2$ as

$$d((a_1, a_2), (b_1, b_2)) = |a_1 - b_1| + |a_2 - b_2|.$$

(You can think of a point as an intersection of streets in a city following a grid plan, and the distance is measured along the streets, with one block counting as one unit in either horizontal or vertical direction). We then define an equivalence relation R in $\mathbb{Z}$ by saying that $R(a, b)$ if and only if $d((0, 0), a) = d((0, 0), b)$. What are the equivalence classes? Visualize.

We usually think of square-shaped city blocks here. In reality the typical blocks on Manhattan are not squares at all, but rectangles whose "width" is much bigger than their "height".