3B Relations, functions, cardinality

Some terminology: "Injection", "injective function" and "one-to-one function" are synonyms. ("Injection" is a noun, "injective" is an adjective.)

"Surjection", "surjective function" and "onto function" are synonyms. A surjection $f : A \to B$ is said to be "onto B" because it "covers" all of B.

"Bijection", "bijective function" and "one-to-one and onto function" are synonyms.

3B1 (Equivalences and orders) Let $\Omega = \{1, 2, 3\}$. We define two relations over subsets of Ω: let $E(S, T)$ be the relation $|S| = |T|$, and let $R(S, T)$ be the relation $|S| \leq |T|$. Recall that for finite sets, $|\ldots|$ means the number of elements.

Describe the two relations in words. Is E an equivalence relation? If yes, what are the equivalence classes? Is R an order relation? If yes, is it a total order?

3B2 (Dual orders) If R is a relation on set A, its *dual* (or opposite, or converse) relation R^d is the relation defined by $R^d(x, y) \leftrightarrow R(y, x)$ for all $x, y \in A$ (we are viewing relations as predicates here). When using infix notation, a typical convention is to reverse the symbol, for example $\leq^d$ is written $\geq$, and $\subseteq^d$ is written $\supseteq$.

(a) Prove: If R is an order relation, then R^d is an order relation.

(b) Prove: If R is a total order, then R^d is a total order.

(c) If R is an order relation, what kind of relation is $R \wedge R^d$?

(d) What does the result from (c) say specifically for the relations $\leq$ (on real numbers), $\subseteq$ (on subsets of some set), and divisibility $|$ (on positive integers)?

3B3 (Finite domains) Let $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$.

(a) How many different injections are there from A to B? (Hint: You can list them, but you could also just count. What are the possibilities for the value of $f(1)$? If you choose that $f(1)$ has a particular value, how many possibilities are then for the value of $f(2)$?)

(b) How many different surjections are there from A to B?

(c) How many different injections are there from B to A?

(d) How many different surjections are there from B to A?

(e) How many different bijections are there from A to A?

3B4 (Infinite domains) Give an example of a function $\mathbb{N} \to \mathbb{N}$ that is

(a) injective but not surjective,

(b) surjective but not injective,

(c) bijective (that is, injective and surjective), *other than* the identity function $f(x) = x$,

(d) neither injective nor surjective.

Hint: Very simple functions should suffice in all cases. You don't need to construct very complicated ones. Try, for example, simple arithmetic (e.g. adding or subtracting a constant, or multiplying), or a constant function, or a function defined by cases. But make sure that your function is a well-defined function from $\mathbb{N}$ to $\mathbb{N}$, that is, for every $x \in \mathbb{N}$ you have $f(x) \in \mathbb{N}$.

3B5 (Function composition) Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove: If f and g are injections, then $g \circ f$ is an injection.

(b) Prove: If f and g are surjections, then $g \circ f$ is a surjection.

(c) Prove: If f and g are bijections, then $g \circ f$ is an bijection.

Hint: For injection, it is perhaps easiest to work with the contrapositive form: f is injective if $x \neq y \Rightarrow f(x) \neq f(y)$.

Part (c) shows that our definition of "having same cardinality", which is based on the existence of a bijection, is a *transitive* relation between sets, even for infinite sets: if A, B have the same cardinality, and B, C have the same cardinality, then so do A, C.

3B6 (Cardinality of power sets) Prove, by explicitly constructing an injection, that if A is any set (finite or infinite), then $|A| \leq |P(A)|$, where $P(A)$ means the power set. Hint: There is a very simple injection.

For finite sets we know that if $|A| = n$, then $|P(A)| = 2^n$, so this exercise proves, in a set-theoretic way, that $2^n \geq n$ for all $n \in \mathbb{N}$. Of course this inequality could be proved in other ways, for example by induction.

3B7 (Cardinality of intervals) Recall that $[a, b]$ and $]a, b[$ denote closed and open intervals of real numbers. We say that two sets A and B are *equipotent*, or "have the same cardinality", if there exists a bijection between them.

(a) Prove, by constructing a bijection, that every nonempty closed interval $[a, b]$ (where $a < b$) is equipotent to the closed unit interval $[0, 1]$. Hint: It is enough to construct a bijection between the two sets in one direction. Which direction is easier?

(b) Prove, by constructing a bijection, that every nonempty open interval $]a, b[$ (where $a < b$) is equipotent to the open unit interval $]0, 1[$.

(c) From the previous parts, conclude that *all* nonempty closed intervals have the same cardinality, and *all* nonempty open intervals have the same cardinality.

(d) Construct an injection from the open $]0, 1[$ to the closed $[0, 1]$.

(e) Construct an injection from the closed $[0, 1]$ to the open $]0, 1[$.

It is a general fact (but not quite trivial to prove) that if there are injections both ways $f : A \to B$ and $g : B \to A$, then there is also a bijection $h : A \to B$. This is known as the Schröder–Bernstein theorem. We will not prove this on the course (but look up the proof if you are interested). From (d) and (e), and applying the Schröder–Bernstein theorem, we can conclude that the closed interval $[0, 1]$ and the open interval $]0, 1[$ are equipotent, so there *exists* a bijection between them. One could also construct an explicitly defined bijection between them (but it is not quite straightforward).

The challenge problem is worth an extra point.

3B8 (** CHALLENGE: Cardinality of powersets, part II) Prove that if A is a set (either finite or infinite), there is no bijection from A to its powerset $P(A)$.

Hint: It is enough to show that there is no surjection. Suppose that f is a function from A to $P(A)$. For each element $x \in A$, either $x \in f(x)$ or not. Now consider the set

$$T = \{x \in A \ : \ x \notin f(x)\}.$$

Prove that T is not equal to $f(y)$ for any $y \in A$. That means that f is not a surjection to $P(A)$, because there is an element $T \in P(A)$ not covered by f.

Together with 3B6, this exercise shows that for any set A, the powerset $P(A)$ has different (bigger) cardinality than A. For example, because we can form power sets of power sets,

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < \dots$$

In particular, there are *infinitely many* cardinalities that infinite sets can have, not just two (“same as $\mathbb{N}$” and “same as $\mathbb{R}$”).