4B Enumerative combinatorics II

4B1 (Higher derivatives) Recall that the (first) derivative of x^n (with respect to x) is nx^{n-1}, and the $(k+1)$th derivative is the derivative of the kth derivative.

(a) What are the second, 3rd, 4th, 5th and 6th derivatives of x^5?

(b) Generally, if $1 \leq k < n$, what is a simple expression for the kth derivative of x^n, expressed using the falling product?

(c) What is a simple expression for the nth derivative of x^n, expressed using the factorial?

(d) What is the kth derivative of x^n when $k > n$?

4B2 (Inclusion-exclusion) A DNA sequence is a string (tuple) of letters, each chosen from the set $\{A, C, G, T\}$. For example, $TAGGA$ is a possible sequence of length 5.

(a) How many DNA sequences of length 5 exist that contain each of the letters A, C, G at least once? (T can also appear, but is not required to.)

(b) (OPTIONAL, not required for scoring the problem, and no extra points.) How many DNA sequences of length 5 exist that contain each of the letters A, C, G, T at least once?

4B3 (Round table) Six guests ABCDEF are to be seated around a round table. The host wants to list all possible arrangements (and then ponder which arrangement is the best for good discussions). What matters is who sits as the left neighbor of whom, and who sits as the right neighbor of whom. If an arrangement is rotated clockwise or counterclockwise, the host considers it the same arrangement (listed only once).

Guests A and B cannot stand each other, so arrangements that seat them as neighbors are not possible. Count the possible arrangements.

Hint: The number is almost a hundred, so it is probably best for *you* not to list all arrangements. But you could list some "partial" arrangements (make some choices concerning one or two guests), and then count the remaining choices by arithmetic.

4B4 (Back to the lab) We are back in the metallurgic laboratory (cf. 4A1). Now we are considering mixtures of k substances (we think of iron as just one possible substance). We measure the amount of each substance in *tenths* of the total mass, and we only consider integer numbers of the tenths. For each substance, any number from 0 to 10 is possible, but they must add up to ten: for example, if given three substances, some of their possible mixtures are $(5, 4, 1)$, $(10, 0, 0)$ and $(3, 3, 4)$.

(a) We are considering two substances A and B. What are their possible mixtures, and how many are they?

(b) We are considering three substances A, B and C. How many possible mixtures are there? (Hint: Consecutive choice. If you have chosen the amount of A to be x tenths, how many choices do you have for the amount of B?)

(c) We are considering k substances $A_1, A_2, \ldots, A_k$. Find a general formula, involving binomial coefficients, for the number of mixtures these substances. (Hint: Stars and bars from lecture 7, or lecture notes §2.2.2)

(d) Apply your general formula to the cases $k = 1, 2, 3, 4$. Does the result for $k = 1$ make sense? Do the results for 2 and 3 match what you calculated in (a)–(b)?

4B5 (Discrete maximization) Consider the function $f : \{0, 1, 2, \ldots, n\} \to \mathbb{R}$,

$$f(k) = \binom{n}{k} p^k q^{n-k},$$

where $n = 50$, $p = 0.2$ and $q = 0.8$ are given constants. We want to find the integer k where $f(k)$ attains its maximum. If f were a continuously differentiable function from real numbers, we would probably begin by finding where its derivative is zero. But our f is defined only on integer values of k. So we will do something analogous in our discrete situation.

(a) For an arbitrary $k \in \{0, 1, 2, \ldots, n-1\}$, express the *quotient*

$$Q(k) = \frac{f(k+1)}{f(k)}$$

in a simple form. Hint: In 4A4 you have already worked out the quotient $\binom{n}{k+1} / \binom{n}{k}$. Now you just have some extra factors, and many of them should cancel out. Observe that $Q(k) > 1$ means that $f(k+1) > f(k)$, and similarly for "=" and "<".

(b) Using your formula from (a), find when $Q(k) > 1$. You should be able to find a value k^* such that for all $k < k^*$ we have $Q(k) > 1$, and for all $k \geq k^*$ we have $Q(k) < 1$.

Hint: You can solve the inequality $Q(k) > 1$ algebraically, rearranging it so that k is on one side and everything else is on the other side. Remember that n, p, q are constants. If it helps, you can begin by studing some numerical values of $Q(k)$ in the interval $7 \leq k \leq 13$, to get a feeling of how the function behaves.)

(c) Can you now say that $f(k^*)$ is the largest value that f takes? Why?

(d) Calculate the three values $f(k^* - 1), f(k^*), f(k^* + 1)$, verifying that at least locally $f(k^*)$ is indeed a maximum, that is $f(k^* - 1) < f(k^*) > f(k^* + 1)$.

In this exercise we have learned a method for finding the maximum of a function whose domain is discrete. Finding k^* is analogous to finding a zero of the derivative: it is the point where f turns from increasing to decreasing.

The same method can be applied generally, with other functions. An alternative would be to study the *differences* $f(k+1) - f(k)$, but in combinatorics, quotients often have nicer form.

For those who already know something about probabilities: The function f is the so-called *probability mass function* of a binomial distribution. Suppose we have a biased coin that has probability $p = 0.2$ for "heads", and probability $q = 0.8$ for "tails" whenever we toss it. If we toss it $n = 50$ times, then $f(k)$ is the probability that we obtain exactly k "heads" (and $n - k$ "tails"). In this exercise we have found, by combinatorial calculations, the most probable number of heads (k^*) and its probability ($f(k^*)$).

4B6 (Twelve days of Christmas) In a popular Christmas song the protagonist receives increasing numbers of gifts every day. Here are the first five days. In the song it continues in the same manner for twelve days.

On the first day of Christmas, my true love gave to me
    A partridge in a pear tree.

On the second day of Christmas, my true love gave to me
    Two turtle doves,
    And a partridge in a pear tree.

On the third day of Christmas, my true love gave to me
    Three French hens,
    Two turtle doves,
    And a partridge in a pear tree.

On the fourth day of Christmas, my true love gave to me
    Four calling birds,
    Three French hens,
    Two turtle doves,
    And a partridge in a pear tree.

On the fifth day of Christmas, my true love gave to me
    Five golden rings,
    Four calling birds,
    Three French hens,
    Two turtle doves,
    And a partridge in a pear tree.

(a) How many gift items are given on days 1, 2, 3, 4 and 5? (Just add them up. E.g. three French hens are *three* items. A partridge is one item. Don't count the pear tree: it is not a gift, the partridge is just sitting there.)

(b) Generally, how many gift items are given on the kth day, when k is any positive integer? Assume that the number of gifts keeps increasing in the same manner. Start by writing it as a big sum (using $\sum$), but then give a general formula using less than ten basic arithmetic operations (no big sum). Hint: Lecture 6.

(c) Using the numbers from (a), calculate the *running totals* (or *cumulative sums*) on days 1 to 5. That is, on each day, calculate how many items have been given so far (including that day). Just the numbers, no general formula required here. Note that each day a *new* partridge is given, and two *new* turtle doves etc.

(d) Go to https://oeis.org/, enter your numbers from (c), and hit Search. From the results, find how many items are given over the twelve days, in total. Search for "Christmas" on the result page to see if it is mentioned.

(e) (OPTIONAL – Not required for scoring the problem, and no extra points.) OEIS gives a simple general formula for the total number of items given during the first n days. Try to prove, e.g. by induction, that the formula is correct.