6A Graphs / Number theory

6A1 (Metric) A *distance function*, or a *metric*, in a set S is a function $d : (S \times S) \to \mathbb{R}$ such that for all $x, y, z \in S$:

(i) $d(x, y) \geq 0$ (nonnegativity)

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$ (symmetry)

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Now we study distances between vertices of a graph.

(a) Prove or disprove: If $G = (V, E)$ is a connected graph, the following function ℓ is a metric in V:

$$\ell(x, y) = \text{the length of the shortest path between } x \text{ and } y$$

(b) Prove or disprove: If $G = (V, E)$ is a connected graph, the following function m is a metric in V:

$$m(x, y) = \ell(x, y) + 1$$

(Note: $m(x, y)$ is the number of vertices in a shortest path between x and y.)

(c) The *diameter* of a connected graph is the largest distance that can be found between any two vertices. What are the diameters of K_n (complete graph, $n \geq 1$), C_n (cycle, $n \geq 3$) and S_n (star graph, $n \geq 2$), when using ℓ as the distance function?

(d) List the diameters of C_n for $3 \leq n \leq 9$ and enter them to `https://oeis.org/`. How many different sequences are found? Look at the first search result. Do you think this is the diameter of C_n, looking at (i) its description and (ii) how the sequence continues?

6A2 (Disconnected graphs)

(a) Prove or disprove: In any graph, connected or disconnected, this function is a metric:

$$c(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \text{ and there is a path from } x \text{ to } y \\ 2 & \text{otherwise} \end{cases}$$

(b) Is ℓ (from problem 1) a well-defined metric in a disconnected graph?

6A3 (Family tree) Let $T_n = (V_n, E_n)$ be the (ancestral) family tree of a fixed person c, going back n generations ($n \geq 0$). T_0 has c as its only vertex. T_1 contains also the two parents of c, with edges from c to both. T_2 adds the four grandparents and so on.

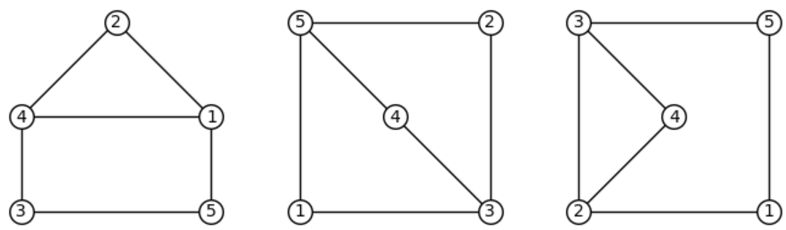
(a) Find $|V_n|$, $|E_n|$, $\chi(T_n)$, and the diameter of T_n.

(b) Find the *average distance* from c to all people in the family tree,

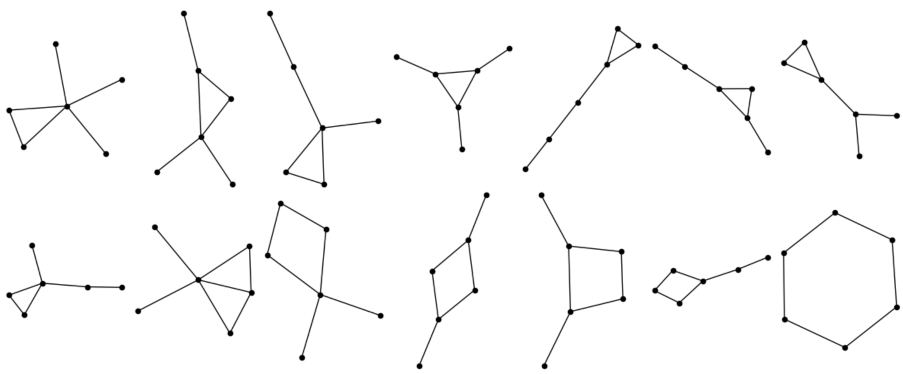
$$a_n = \frac{1}{|V_n|} \sum_{x \in V_n} \ell(c, x),$$

when $n = 10$. Give the result in decimal form with three decimals.

6A4 (Isomorphic or not) Consider the following three graphs. Find the degrees of each of their vertices. Find which graphs (if any) are isomorphic, and construct an isomorphism (function) between them. If two graphs are not isomorphic, explain why it is impossible to construct an isomorphism between them.



6A5 (List correctness) A graph is *unicyclic* if it contains exactly one cycle graph (as a subgraph). Lecturer K. has constructed what he thinks is a complete listing of all unlabeled connected unicyclic graphs of six vertices.



As his research assistant you have learned to suspect his listings. You check https://oeis.org/A001429, and you find that the correct count of such graphs is 13, while K's list contains 14 graphs. Something is wrong. (Of course it *could* be the OEIS entry, but let's assume it is correct.)

(a) Find one graph that should not be on the list at all.

(b) After removing that graph, the count is now correct. Does this prove that the list is correct?

(c) You suspect that the list has one graph twice (a duplicate). Find it.

(d) You also suspect that one graph is missing from the list. What is it?

6A6 (Divisibility) Recall that for two integers a, b (not necessarily positive!) we say that a *divides* b, or in short $a \mid b$, if there exists an integer m such that $b = ma$.

Prove or disprove each of the following statements. All variables are understood to be integers. For those statements that are true, *also* give a small example with concrete numbers (preferably positive). For the statements that are false, a small counterexample can probably be found.

(a) If $a \mid b$ and $b \mid c$, then $a \mid c$.

(b) If $a \mid b$, then $b \mid a$.

(c) If $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

(d) If $a \mid b$ and $a \mid c$, then $a \mid b + c$.[1]

(e) If $a \mid b$ and $a \mid b + c$, then $a \mid c$.

(f) If $a \mid b$, and c is any integer, then $a \mid bc$.

(g) If $a \mid c$ and $b \mid c$, then $ab \mid c$.

(h) If $a \mid c$ and $b \mid c$, then $ab \mid c^2$.

(i) If $a \mid c$ and $b \mid d$, then $ab \mid cd$.

(j) If $a = bc > 0$, then $b \leq \sqrt{a}$ or $c \leq \sqrt{a}$ (or both).

6A7 (Simple Diophantus) Consider this equation in two variables.

$$5x - 2y = 1$$

By high-school methods you can easily find an infinite number of solutions $(x, y) \in \mathbb{R}^2$, indeed for any $x \in \mathbb{R}$ you could find a suitable $y \in \mathbb{R}$. However, we are now looking for integer solutions $(x, y) \in \mathbb{Z}^2$.

(a) Find one solution $(x, y) \in \mathbb{Z}^2$. (Hint: Trial and error with small positive integers should be enough.)

[1] You should read "$\mid$" as a relation symbol like "$=$" and "$<$", so it binds more loosely than any arithmetic operation. That is, $a \mid b + c$ means $a \mid (b + c)$.

(b) From your one solution, construct another (x', y'). (Hint: If x increases by some amount c, how must y change so that the equality stays? Make it so that the new values are again integers.)

(c) Construct an infinite set of integer solutions (x, y) to the equation. (You do not need to prove that your construction contains *all* possible solutions, but at least it should contain infinitely many.)

Equations like this, where only integer solutions are desired, are called *Diophantine equations*, after Diophantus of Alexandria who lived in the 3rd century AD. His book *Arithmetica* contains many such problems.

6A8 (** CHALLENGE, worth an extra point: Variant of geometric series)

In discrete mathematics an often recurring task is computing sums.[2] One example is the *arithmetico-geometric sum*

$$s_n = \sum_{k=1}^{n} k a^k,$$

where $a \neq 1$ is a constant. Your task is to find a closed form expression[3] for it. Here are some hints to get started:

- Write out as_n as a sum.
- Write out $s_n - as_n$ as a sum, collecting like terms from s_n and as_n (terms having the same power of a).
- Solve for s_n, simplify, and recognize a part of your expression as something you know already.

Once you have your general closed form expression for s_n with any $a \neq 1$, write a specific form for s_n when $a = 2$, and try to simplify it further. Then verify it against explicit summation with $n = 1, 2, 3, 4$. Then calculate 3b again with your new shiny formula.

Calculate also s_n with $a = 10$ and $n = 5$ and verify against direct summation.

Finally, think why we had to assume $a \neq 1$ for our general method. If $a = 1$, do you know a simpler method?

[2] Somewhat analogous to integrals in calculus — and sometimes quite as difficult.

[3] An expression containing a fixed, finite number of elementary arithmetic operations — in particular, not using the big sum symbol $\sum$ or three dots.