1A Introductory exercises

This first set contains problems to be solved during the first exercise session (1A). You obtain exercise points for the session by attending the session.

Sets

Recall the notion of a *set* as an unordered collection of distinct elements. Check lecture notes pages 3—10 as needed. Recall that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the sets of natural numbers (that is, non-negative integers), integers, rational numbers, and real numbers. Note that our natural numbers begin from zero (some people use a different definition where they begin from one).

The *set builder notation* (page 4) is a method for specifying a set without listing its elements explicitly. For example,

$$\{x \in \mathbb{Z} \ : \ 4 \leq x \leq 7\}$$

means (by definition) the set of those integers whose value is at least 4 but no more than 7, that is,

$$\{4, 5, 6, 7\}.$$

1A1 (Intervals of integers) How many elements do the following sets contain? Here a and b are some arbitrary integers (possibly $a < b$ but possibly not). Try to make the resulting expression as simple as possible.

(a) $\{x \in \mathbb{Z} \ : \ 10 \leq x \leq 15\}$ (Hint: You can start by listing the elements.)

(b) $\{x \in \mathbb{Z} \ : \ 10 < x < 15\}$

(c) $\{x \in \mathbb{Z} \ : \ 10 \leq x < 15\}$

(d) $\{x \in \mathbb{Z} \ : \ 20 \leq x \leq 15\}$

(e) $\{x \in \mathbb{Z} \ : \ a \leq x \leq b\}$

(f) $\{x \in \mathbb{Z} \ : \ a < x < b\}$

(g) $\{x \in \mathbb{Z} \ : \ a \leq x < b\}$

Solution.

(a) 6, because the set is $\{10, 11, 12, 13, 14, 15\}$.

(b) 4, because the set is $\{11, 12, 13, 14\}$.

(c) 5, because the set is $\{10, 11, 12, 13, 14\}$.

(d) 0, because the set is empty (there are no numbers that fulfil the given condition).

(e) $b - a + 1$, but if this number would be negative, then zero.

(f) $b - a - 1$, but if this number would be negative, then zero.

(g) $b - a$, but if this number would be negative, then zero.

People are sometimes surprised by these counts. When counting elements in discrete sets, defined by some endpoints, one must be careful about whether the endpoints are included in the set or not, otherwise one may make an *off-by-one error* or *fencepost error*. Note that if you build a straight fence that is 10 meters long and has 1-meter segments, you need 11 fenceposts! (`https://en.wikipedia.org/wiki/Off-by-one_error`)

A concise way of saying "$b - a + 1$, but if this would be negative, then zero" is

$$\max\{b - a + 1,\ 0\},$$

that is, choose the bigger of the two quantities $b - a + 1$ and 0.

1A2 (Ellipsis and counting) The *ellipsis notation* is another handy way of writing a large set, but it is also dangerous. The meaning of the three dots is supposed to be "continuing in the same manner", and it is up to the reader to understand what is "the same manner". Easy examples are some consecutive integers like $\{5, 6, 7, \ldots, 12\}$ and some consecutive even integers like $\{2, 4, 6, 8, \ldots, 20\}$.

Count the elements of the following sets. List their elements if convenient. "Counting" means "determining the number of something" (not necessarily by listing one by one).

(a) $\{10, 12, 14, \ldots, 20\}$

(b) $\{0, 2, 4, \ldots, 100\}$

(c) $\{1, 4, 9, \ldots, 100\}$ (meaning: squares of consecutive integers)

(d) $\{1, 4, 9, \ldots, 1\ 000\ 000\}$

(e) $\{1, 4, 9, \ldots, a\}$, where a is a square of an integer.

Solution.

(a) 6. The set is, in explicit notation, $\{10, 12, 14, 16, 18, 20\}$.

(b) 51. Beware of the fencepost error.

(c) 10. The set is $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$.

(d) 1000. Note that the square roots of the elements are simply the integers $\{0, 1, 2, \ldots, 1000\}$ and they are easy to count. Each of these integers, when squared, gives a square that is between 0 and 1 000 000 (endpoints included). — It would be very inconvenient to go through the integers from 0 to one million and check, one by one, which ones happen to be squares.

(e) $\sqrt{a}$. The square roots of the elements are $1, 2, \ldots, \sqrt{a}$.

1A3 (Operations of sets) Recall the notions of closed, open and half-open intervals (of real numbers).

$$\begin{aligned} [a,b] &= \{x \in \mathbb{R} \ : \ a \le x \le b\} \\ ]a,b[&= \{x \in \mathbb{R} \ : \ a < x < b\} \\ [a,b[&= \{x \in \mathbb{R} \ : \ a \le x < b\} \\ ]a,b] &= \{x \in \mathbb{R} \ : \ a < x \le b\} \end{aligned}$$

Caveat: Some people use the notation (a, b) for the open interval (this could be confused with the similar notation for an ordered pair of two numbers).

In the following, a, b, c, d are real numbers.

(a) What is $[a, a]$? How many elements does it contain?

(b) What is $]a, a[$? How many elements does it contain?

(c) What is $[a, b]$ if $b < a$?

(d) What is $[5, 10] \cup [7, 12]$? Can it be expressed as an interval?

(e) What is $[5, 10] \cap [7, 12]$? Can it be expressed as an interval?

(f) What is $[5, 10] \cup [15, 20]$? Can it be expressed as an interval?

(g) What is $[5, 10] \cap [15, 20]$? Can it be expressed as an interval?

(h) When exactly is $[a, b] \cup [c, d]$ an interval?

(i) When exactly is $[a, b] \cap [c, d]$ an interval?

(j) What is $[0, 10] \setminus [4, 6]$? (Note the set difference operation.) Express it as an union of two intervals, be careful with the endpoints.

Solution.

(a) The set $\{a\}$, which contains one element.

(b) The empty set $\{\}$, which contains zero elements.

(c) Again the empty set.

(d) $[5, 12]$

(e) $[7, 10]$

(f) This is a subset of the real numbers, consisting of two disjoint intervals: the real numbers from 5 to 10 (endpoints included) and the real numbers from 15 to 20 (endpoints included). It is not an interval.

(g) Empty set. It can be expressed as an interval, e.g. $]4,4[$ if one likes.

(h) One way to express the condition is: If the at least one of the endpoints of the second interval, that is c or d, is in the first interval, then the union of the two intervals is again an interval. In other words: $a \leq c \leq b$ or $a \leq d \leq b$. (If this is not the case, then the two intervals are disjoint, and there is "empty space" between them.)

(i) The intersection of two intervals is always an interval (possibly empty).

(j) $[0,4[\, \cup \,]6,10]$.

1A4 (Set membership and logic) A *statement* is a claim that is either true or false. If it contains variables (like x), its truth value may depend on the value of x: for example, $x < 3$ may be true or false depending on what x is. Typical mathematical statements are equality ($x = 4$), inequalities ($x < 4$), and set membership ($x \in \{2,4,6\}$).

Statements can also be combined into bigger statements by *connectives*. The simplest connectives are $\wedge$ (**and**, meaning that both components are true) and $\vee$ (**or**, meaning that at least one of the components is true — possibly both). To aid your memory, note that $\wedge$ looks a little like an A ("and").[1]

Now we explore the connection between connectives and set membership. Let[2]

$$\begin{aligned} A &= \{1,2,3,\ldots,10\} \\ B &= \{x \in A \ : \ x \leq 5\} \\ E &= \{x \in A \ : \ x \text{ is even}\} \end{aligned}$$

(a) Write out the sets B and E explicitly or using the ellipsis notation.

(b) Write out the intersection $B \cap E$.

(c) Write out the union $B \cup E$.

(d) Consider the "and" (conjunction) statement

$$(x \leq 5) \ \wedge \ (x \text{ is even}).$$

For which of the values $x \in A$ is this statement true? (Hint: Consider individual values of x. For example, if $x = 1$, is the statement true of false? Then what if $x = 2$?) Is this equal to one of sets seen in (b) and (c)?

[1]Those who excel in Latin may want to remember that *vel* means "or", and $\vee$ looks like a V. Finns might recall that *vai* is a bit like *tai*.

[2]An integer is *even* if its value is two times some integer, e.g. 8 is even because $8 = 2 \times 4$. Otherwise it is *odd*. For example, $0, 2, 4, 6$ are even and $1, 3, 5, 7$ are odd.

(e) Consider the "or" (disjunction) statement

$$(x \leq 5) \;\vee\; (x \text{ is even}).$$

For which of the values $x \in A$ is this statement true? Is this equal to one of sets seen in (b) and (c)?

(f) Do you notice some useful connection between the symbols $\cap, \cup, \wedge, \vee$?

Solution.

(a) $\{1, 2, 3, 4, 5\}$

(b) $\{2, 4, 6, 8, 10\}$

(c) $\{2, 4\}$

(d) $\{1, 2, 3, 4, 5, 6, 8, 10\}$

(e) For example, when $x = 1$, the statement "x is even" is false, so the conjunction is also false. However, when $x = 2$, both components are true, so the conjunction is true. The values that make it true are $2, 4$, exactly the set in (b).

(f) For example, when $x = 1$, the statement "x is even" is false, but the statement "$x \leq 5$" is true. Having one of the components true is enough to make the disjunction true. On the other hand, when $x = 7$, neither component is true, so the disjunction is false. The values that make the disjunction true are $1, 2, 3, 4, 5, 6, 8, 10$, exactly the set in (c).

(g) Intersection ($\cap$) and conjunction ($\wedge$) are paired in this sense: belonging to the intersection, $x \in (B \cap E)$, is the same thing as belonging to both sets: $(x \in B) \wedge (x \in E)$. We observe the graphical similarity of $\cap$ and $\wedge$.

Similarly, union and disjunction are paired: belonging to the union, $x \in (B \cup E)$, is the same thing as belonging to at least one of the component sets: $(x \in B) \vee (x \in E)$. We observe that $\cup$ and $\vee$ are similar.

1A5 (Boolean searches) We have a database of a million people. We can think of this as a set A. For each person, the database contains information such as citizenship, which could be Finnish, Swedish, or something else (each person has only one citizenship). The user can form a query like "x is Finnish", and the database system then goes through every person in the database, checks if this statement is true for that person, and returns the list of matching persons. A query can also combine statements using connectives **and** and **or**. The database contains 200 000 Finnish citizens and 100 000 Swedish citizens.

A novice user wants to count all Finns **and** Swedes, and writes the query: "x is Finnish **and** x is Swedish". To the user's utter surprise, the result set is empty (zero persons). Explain.

Solution. The database system looks at one person at a time. This query would return a person x only if that person were both Finnish **and** Swedish, which never happens in this database.

What the user seems to want would be returned by the query "x is Finnish **or** x is Swedish". This would return the union of the two sets.

It is true that the desired result (union) contains all Finns **and** it contains all Swedes, but if we take the viewpoint of an individual (belonging to the result set or not) we need to use the disjunction.

1A6 (Subsets) If A and B are two sets, then A is a *subset* of B (written $A \subseteq B$) if every element of A is also an element of B. For example $\{1,3\} \subseteq \{1,2,3,4\}$. Or, in other words: A does *not* contain any element that would *not* also be in B. Note that A could be empty (indeed, an empty set does not contain any elements that are not in B), and also A and B could be the same set.

(a) List all subsets of $A = \{1,2\}$. (There should be four of them.)

(b) List all subsets of $B = \{1,2,3\}$ using the following method: first list the zero-element subsets, then the one-element subsets, then the two-element subsets, and then the three-element subsets. (Remember that order does not matter in a set: $\{b,a\}$ is the same set as $\{a,b\}$.) How many of each size do you obtain? How many in total?

(c) List all subsets of $B = \{1,2,3\}$ using the following method: Consider, in turn, each of the sets you listed in (a). For each one, either add the element 3 or not. How many different sets do you obtain and why? Does this method necessarily produce all the subsets, and each one exactly once?

(d) List and count all subsets of $C = \{1,2,3,4\}$ using a method similar to (b). Try to be systematic. How do you make sure that you list every possibility, but each one only once?

(e) If you listed all subsets of $C = \{1,2,3,4\}$ using a method similar to (c), that is, taking all the subsets of $\{1,2,3\}$ and then either adding 4 or not, how many would there be? (You can actually do the listing if you want, but it is not necessary.)

(f) What happens to the number of subsets when the original set is grown by one element?

(g) How many different subsets does an n-element set have? Try to write a concise expression. How many different subsets does a 10-element set have?

(h) Try your expression from the previous item for the subsets of a one-element set $\{1\}$, and for a zero-element set.

Solution.

(a) $\varnothing$, $\{1\}$, $\{2\}$ and $\{1,2\}$.

(b) Zero elements: $\varnothing$ (1 subset).
One element: $\{1\}$, $\{2\}$, and $\{3\}$ (3 subsets).
Two elements: $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ (3 subsets).
Three elements: $\{1,2,3\}$ (one subset).
Total $1+3+3+1=8$ subsets.

(c) When we do not add 3, we get the original four

$$\varnothing, \{1\}, \{2\}, \{1,2\},$$

which are obviously also subsets of the larger set B. When we do add 3, we get another four sets

$$\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}.$$

In total we get $4+4=8$ subsets, same as what we got in (b).

We do get every subset of B, because every subset *either* does not contain 3 (so it is listed in the first part) or contains it (in which case it is listed in the second part). With some thinking it should also be clear that every subset is listed only once (for example, that the second part does not list any of the subsets in the first part again).

Here it may seem obvious to you (at least after some thought) that the method is correct, but in general, trying to list and count the members of some collection correctly can be quite difficult — sometimes we make the error of missing some, and sometimes we make the error of counting some members twice.

(d)

0 elements	$\varnothing$	1 subset
1 elements	$\{1\}, \{2\}, \{3\}, \{4\}$	4 subsets
2 elements	$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$	6 subsets
3 elements	$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$	4 subsets
4 elements	$\{1,2,3,4\}$	1 subsets

Total $1+4+6+4+1=16$ subsets.

The most difficult are probably the 2- and 3-element subsets. For 2 elements, we used the method of *consecutive choices*: First choose the *smaller* of the elements: it can be 1, 2 or 3 (not 4, why)? Then, for each such choice, choose next the *bigger* element. (Think about it. How would you implement this in a computer program?)

For 3 elements, one can use a similar method of *three* consecutive choices. Alternatively, one can note that to choose 3 elements from 4, one can equivalently choose *one* element to be *left out*. There are four ways to choose this left-out element.

(e) From the 8 three-element subsets of $\{1,2,3\}$, each one yields 2 subsets of $\{1,2,3,4\}$, namely the one without element 4, and the one with it. So there are $2 \times 8 = 16$ subsets.

This matches the count obtained in (d). This is a very important thing in counting. If the same objects are listed or counted by two different methods, the lists must match (of course they may be in different orders) and the total counts must match. Otherwise there is an error somewhere.

Sometimes one method of counting is much easier than another method. Which method do you think is the easiest here?

(f) It seems that the number always doubles, because from each subset of $\{1, 2, \ldots, n-1\}$, we get two different subsets of $\{1, 2, \ldots, n\}$: one with the element n and one without.

(g) It seems that the number is 2^n. On later lectures and exercises we will see a formal way of proving this ("proof by induction"). Thus, for example, a 10-element set has $2^{10} = 1024$ different subsets. (They could be easily listed by a computer, but manual listing would be quite tedious.)

(h) Our formula claims that the 1-element set has $2^1 = 2$ subsets. This is true: the subsets are $\varnothing$ and $\{1\}$.

Also it claims that an empty set has $2^0 = 1$ subsets. This is also true: the only subset of $\varnothing$ is $\varnothing$.