MS-A0402 Foundations of discrete mathematics
Department of mathematics and systems analysis
Aalto SCI

J Kohonen
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Exercise 2A

2A Elementary logic

2A1 (Multi-way intersections)

(a) Is it true that if $A \cap B$, $B \cap C$ and $A \cap C$ are all nonempty, then $A \cap B \cap C$ is nonempty?

(b) Is it true that if $A \cap B$, $B \cap C$ and $A \cap C$ are all empty, then $A \cap B \cap C$ is empty?

In each case, either prove the claim true, or give a concrete counterexample showing it false. (Hint: Very small sets A, B, C might be sufficient.)

Solution.

(a) No. Counterexample: Let $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{2, 3\}$. Then $A \cap B = \{1\}$, $B \cap C = \{3\}$, and $A \cap C = \{2\}$. But $A \cap B \cap C = \emptyset$.

(b) Yes. We give two simple ways to argue this.

Proof 1: Since $A \cap B = \emptyset$, we have $A \cap B \cap C = (A \cap B) \cap C = \emptyset \cap C = \emptyset$, since the intersection of the empty set with anything is empty.

Proof 2: Observe that $(A \cap B \cap C) \subseteq (A \cap B) = \emptyset$, and the only subset of the empty set is the empty set itself, so $A \cap B \cap C = \emptyset$.

2A2 (Another distributive law) Consider the following distributive law of sets.

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C) \qquad (\dagger)$$

(Note that this is different from the distributive law that was proved on lecture 2.)

(a) Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ and $C = \{3, 6, 7\}$. Calculate the LHS and RHS of (†) and verify that they are the same. Draw a picture of the situation.

(b) Prove this law for all sets A, B, C with a mathematical proof written in good style, with complete sentences that form a solid line of thought. (For example, you can use words like "because", "then we know", and so on. You can also freely use symbols of the set operations.)

Solution.

(a) For LHS, we have $A \cap B = \{3\}$, and hence $(A \cap B) \cup C = \{3, 6, 7\}$. For RHS, we have $A \cup C = \{1, 2, 3, 6, 7\}$ and $B \cup C = \{3, 4, 5, 6, 7\}$, hence $(A \cup C) \cap (B \cup C) = \{3, 6, 7\}$.

(b) To show $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$:

Let $x \in ((A \cap B) \cup C)$. This means $x \in A \cap B$ or $x \in C$.

Suppose the first case $x \in A \cap B$ is true. Since $A \cap B \subseteq A \subseteq A \cup C$, this implies $x \in A \cup C$. Similarly, since $A \cap B \subseteq B \subseteq B \cup C$, this implies $x \in B \cup C$. Together, we have $x \in (A \cup C) \cap (B \cup C)$.

Now suppose the second case $x \in C$ is true. Since $C \subseteq A \cup C$ and $C \subseteq B \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

In either case, we have $x \in (A \cup C) \cap (B \cup C)$, therefore $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$.

To show $(A \cap B) \cup C \supseteq (A \cup C) \cap (B \cup C)$:

Let $x \in (A \cup C) \cap (B \cup C)$. That is, it holds that $x \in A \cup C$ and $x \in B \cup C$. Consider two cases: $x \in C$ and $x \notin C$, where we know that one of them has to be true.

Suppose the first case $x \in C$ is true. Since $C \subseteq (A \cap B) \cup C$, we have $x \in (A \cap B) \cup C$.

Suppose the second case $x \notin C$ is true. Then since $x \in A \cup C$, it must be that $x \in A$. Similarly, since $x \in B \cup C$, it must be that $x \in B$. This implies $x \in A \cap B$. Since $A \cap B \subseteq (A \cap B) \cup C$, we have $x \in (A \cap B) \cup C$.

In either case, we have $x \in (A \cap B) \cup C$, therefore $(A \cap B) \cup C \supseteq (A \cup C) \cap (B \cup C)$.

2A3 (Quantifiers and connectives) Here $P(x)$ and $Q(x)$ are some predicates on natural numbers. Are the two propositions

$$\exists x \in \mathbb{N} : (P(x) \wedge Q(x))$$

and

$$\big(\exists x \in \mathbb{N} : P(x)\big) \wedge \big(\exists x \in \mathbb{N} : Q(x)\big)$$

necessarily equivalent? If yes, prove it. If not, give a counterexample (give some predicates P and Q where the two propositions have different truth values). (Hint: Even if two existential quantifiers use the same symbol x, it need not have the same value in the two.)

Solution. No. Counterexample: Let $P(x) = (x \text{ is odd})$ and $Q(x) = (x \text{ is even})$. Then $\big(\exists x \in \mathbb{N} : P(x)\big)$ is true, for example $x = 1$ is odd, and $\big(\exists x \in \mathbb{N} : Q(x)\big)$ is true, for example $x = 2$ is even. Hence $\big(\exists x \in \mathbb{N} : P(x)\big) \wedge \big(\exists x \in \mathbb{N} : Q(x)\big)$ is true. But $(\exists x \in \mathbb{N} : (P(x) \wedge Q(x)))$ is false, since there is no $x \in \mathbb{N}$ that is both odd and even.

2A4 (Equivalence and implication) Show directly using a truth table (as on lecture 2) that the proposition $p \leftrightarrow q$ always has the same truth value as the proposition $(p \to q) \wedge (q \to p)$.

This means that equivalence $\leftrightarrow$ can be thought of containing "implication in both directions", so it makes sense to use a bidirectional arrow as its symbol.

Solution. The truth table is as follows:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
0	0	1	1	1	1
0	1	0	0	1	0
1	0	0	1	0	0
1	1	1	1	1	1

We observe that the truth values of $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are identical.

2A5 (Proof by cases) A useful method in mathematical proofs is "proof by cases": of two propositions p and q, we somehow already know that at least one is true (that is, $p \vee q$), but we do not know which one. Also, we somehow know that either one would by itself imply another proposition r, which we want to prove. Then we claim that r is indeed true.

Let us try to formalize this rule by using a truth table. That is, show that this proposition is always true:

$$\big((p \vee q) \wedge (p \to r) \wedge (q \to r)\big) \to r$$

(Hint: A truth table of 3 elementary propositions needs $2^3 = 8$ rows.)

Solution. The truth table is as follows:

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to q) \wedge (q \to r)$	$((p \vee q) \wedge (p \to q) \wedge (q \to r)) \to r$
0	0	0	0	1	1	0	1
0	0	1	0	1	1	0	1
0	1	0	1	1	0	0	1
0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

That is, $(p \vee q) \wedge (p \to q) \wedge (q \to r) \to r$ is always true.

2A6 (NOR) Let us consider a new "neither-nor" connective $p \downarrow q$, which says that "neither p nor q is true" (that is, "both are false").

(a) Write the truth table for this connective.

(b) Write a proposition, using only the symbols p, $\downarrow$ and parentheses, that is equivalent to $\neg p$. (Hint: You are allowed to use any of the symbols several times, if you want.)

(c) Write a proposition, using only the symbols p, q, $\downarrow$ and parentheses, that is equivalent to $p \vee q$.

(d) Write a proposition, using only the symbols p, q, $\downarrow$ and parentheses, that is equivalent to $p \wedge q$.

This seems to indicate that we could get rid of our three connectives $\neg, \vee, \wedge$ and just write all our logical propositions using this one symbol instead. (But it might not be very convenient.)

Solution.

(a) The truth table is as follows:

p	q	$p \downarrow q$
0	0	1
0	1	0
1	0	0
1	1	0

(b) $p \downarrow p$. From the truth table in Part (a) we see that if p is false then $p \downarrow p$ is true, and if p is true then $p \downarrow p$ is false. This is identical to $\neg p$.

(c) $(p \downarrow q) \downarrow (p \downarrow q)$. This can be verified by the truth table below:

p	q	$p \downarrow q$	$(p \downarrow q) \downarrow (p \downarrow q)$	$(p \vee q)$
0	0	1	0	0
0	1	0	1	1
1	0	0	1	1
1	1	0	1	1

where we see that the truth values of $(p \downarrow q) \downarrow (p \downarrow q)$ and $(p \vee q)$ are identical.

(d) $(p \downarrow p) \downarrow (q \downarrow q)$. This can be verified by the truth table below:

p	q	$p \downarrow p$	$(q \downarrow q)$	$(p \downarrow p) \downarrow (q \downarrow q)$	$(p \wedge q)$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	1	1

where we see that the truth values of $(p \downarrow p) \downarrow (q \downarrow q)$ and $(p \wedge q)$ are identical.

2A7 (Quantifiers over integers) All of the following universally quantified statements are actually *false*. For each one, find a counterexample x showing this fact.

(a) $\forall x \in \mathbb{Z} : \exists y \in \mathbb{Z} : x = 1/y$

(b) $\forall x \in \mathbb{Z} : \exists y \in \mathbb{Z} : y^2 - x < 100$

(c) $\forall x \in \mathbb{Z} : \forall y \in \mathbb{Z} : x^2 \neq y^3$

Solution.

(a) When $x = 2 = 1/(1/2)$, there does not exist $y \in \mathbb{Z}$ satisfying $x = 1/y$.

(b) When $x = -101$, there does not exist $y \in \mathbb{Z}$ satisfying $y^2 + 101 < 100$, since $y^2 \geq 0$ for any $y \in \mathbb{Z}$.

(c) When $x = y = 1$, we have $x^2 = y^3 = 1$.

Again the "challenge" problem is worth an extra point.

2A8 (** CHALLENGE. Knowledge from ignorance) When three professors are seated in a restaurant, the waiter asks them: "Do you all want coffee?" The first professor says "I do not know." The second professor then says "I do not know." Finally, the third professor says "No, not everyone wants coffee." The waiter comes back and gives coffee to those who want it. How did the waiter figure out who wants coffee and who does not? And who are they?

Solution. Suppose the first professor does not want a coffee, this implies not all professors want a coffee, and his answer would be "no". Since his answer is not "no", we conclude the first professor wants a coffee. Similarly, suppose the second professor does not want a coffee, this implies not all professors want a coffee, and his answer would be "no". Since his answer is not "no", we conclude the second professor also wants a coffee. Finally, suppose the third professor wants a coffee, together with the above, this would mean all three professors want a coffee, and his answer would be "yes". Since his answer is not "yes", we conclude the third professor does not want a coffee. Putting together, the first two professors wants a coffee, and the third does not.