3A Relations

3A1 (Parity arithmetic) The *parity* of an integer means whether it is even or odd. We say that an integer n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$, and *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

For simplicity, we assume now (without actually proving it) that every integer is either even or odd, but not both. (We may return to this topic later on the course.)

Find the parities of $a+b$ and ab in each of these cases, when a and b are arbitrary integers. You can start by calculating some examples (if you need), but you should then prove that your claims hold for *all* integers. (Hint: Induction is probably not needed. Clever use of quantifiers should be enough.)

(a) a is even and b is even

(b) a is even and b is odd

(c) a is odd and b is even

(d) a is odd and b is odd

Is it possible that $a + b$ and ab are both odd? Is it possible that both are even?

Solution.

(a) Let $a = 2k$ and $b = 2h$ for some $k, h \in \mathbb{Z}$.
We have:

$a + b = 2k + 2h = 2(k + h) = 2q$ for some $q \in \mathbb{Z}, q = h + k$
$\Rightarrow a + b$ is even

$ab = 2k(2h) = 2(2kh) = 2s$ for some $s \in \mathbb{Z}, s = 2hk$
$\Rightarrow ab$ is even

(b) Let $a = 2k$ and $b = 2h + 1$ for some $k, h \in \mathbb{Z}$.
We have:

$a + b = 2k + 2h + 1 = 2(k + h) + 1 = 2q + 1$ for some $q \in \mathbb{Z}, q = h + k$
$\Rightarrow a + b$ is odd

$ab = 2k(2h + 1) = 2(2kh + k) = 2s$ for some $s \in \mathbb{Z}, s = 2hk + k$
$\Rightarrow ab$ is even

(c) Same as the part above.

(d) Let $a = 2k + 1$ and $b = 2h + 1$ for some $k, h \in \mathbb{Z}$.
We have:

$a + b = 2k + 1 + 2h + 1 = 2(k + h + 1) = 2q$ for some $q \in \mathbb{Z}, q = h + k + 1$
$\Rightarrow a + b$ is even

$ab = (2k+1)(2h+1) = 4kh+2k+2h+1 = 2s+1$ for some $s \in \mathbb{Z}, s = 2kh+k+h$
$\Rightarrow ab$ is odd

Conclusion: It is not possible that $a + b$ and ab are both odd; but it is possible that both are even, which happens when both a and b are even.

3A2 (Combination of equivalences)

(a) Prove or disprove: if R and S are equivalence relations, then $R \wedge S$ is an equivalence relation.

(b) Prove or disprove: if R and S are equivalence relations, then $R \vee S$ is an equivalence relation.

Solution. Recall: A relation is equivalence if it is reflexive, symmetric, and transitive.

(a) Let A be the set where R and S are equivalence relations of. Let an element $a \in A$. Since R and S are equivalence relations, they are reflexive

$\Rightarrow (a, a) \in R$ and $(a, a) \in S$
$\Rightarrow (a, a) \in R \wedge S \Rightarrow R \wedge S$ is reflexive.

Let $(a, b) \in R \wedge S$
$\Rightarrow (a, b) \in R$ and $(a, b) \in S$
$\Rightarrow (b, a) \in R$ and $(b, a) \in S$, since R and S are symmetric
$\Rightarrow (b, a) \in R \wedge S \Rightarrow R \wedge S$ is symmetric.

Let $(a, b), (b, c) \in R \wedge S$
$\Rightarrow (a, b), (b, c) \in R \Rightarrow (a, c) \in R$ (R is transitive)
$\Rightarrow (a, b), (b, c) \in S \Rightarrow (a, c) \in s$ (S is transitive)
$\Rightarrow (a, c) \in R \wedge S \Rightarrow R \wedge S$ is transitive.

Hence, we have $R \wedge S$ as reflexive, symmetric, and transitive. We can now conclude that $R \wedge S$ is equivalence.

(b) (giving a counter example)
Let $A = \{1, 2, 3\}$ be the set where R and S are equivalence relations of.

Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
$S = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$
$\Rightarrow R \vee S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$
We have $(1, 2) \in R \vee S$ and $(2, 3) \in R \vee S$; however, $(1, 3) \notin R \vee S$
$\Rightarrow R \vee S$ is not transitive
$\Rightarrow R \vee S$ is not equivalence.

3A3 (Bit strings) An n-fold Cartesian product of a set S by itself can be denoted as

$$S^n = S \times S \times \ldots \times S.$$

In particular, $\{0,1\}^n$ is the set of all strings (tuples) of n bits. (A *bit* is one of the two integers 0 and 1.) We define a relation R in $\{0,1\}^n$ such that $R(x,y)$ holds if and only if the strings x and y contain the same number of ones.

(a) Prove that R is an equivalence.

(b) Let $\mathbf{0} = 00\ldots0$ be the string of n zeros. What is the equivalence class $[\mathbf{0}]$?

(c) How many equivalence classes does R have?

Solution.

(a) For any string a in $\{0,1\}^n$, $R(a,a)$ must hold, as any string would have the same number of one as itself.
$\Rightarrow R$ is reflexive

If $R(a,b)$ holds, then $R(b,a)$ must also holds for any $a, b \in \{0,1\}^n$. This is because if a and b have the same number of ones, then b and a will also undoubtedly have the same number of ones.
$\Rightarrow R$ is symmetric

If $R(a,b), R(b,c)$ hold, then $R(a,c)$ must also hold for $a, b, c \in \{0,1\}^n$. This is because if a and b have the same number of ones, b and c have the same number of ones, then a and c must have the same number of ones.
$\Rightarrow R$ is transitive

Hence, R is an equivalence relation.

(b) The equivalence class $[\mathbf{0}]$ with respect to R is the set of all strings that have the same number of ones as the string $\mathbf{0} = 0000...0$ (n number of zeros) in set $\{0,1\}^n$. Since the string $\mathbf{0}$ contains all zeros, any string in the equivalence class must also consist of all zeros. There is only one such string, namely $\mathbf{0}$ itself, so the equivalence class is just the set containing that one element. That is the class is $\{\mathbf{0}\}$.

(c) Recall: Every equivalence relation on A divides A into disjoint equivalence classes of elements that are in the equivalence relation with each other - from lecture notes

In the case of R, each equivalence class will correspond to a different number of ones in the strings. There are n bits in a string; hence, the number of ones ranges from 0 to n, 0 and n inclusive.

$\Rightarrow$ There are $n + 1$ equivalence classes for R

3A4 (Divisibility) When a and b are positive integers, we say that a *divides* b (written $a \mid b$) if there is an integer k such that $b = ka$.

Prove that $\mid$ is an order, but not a total order, on positive integers.

You may notice that our definition really talks about multiplication, not about division. We'll talk more about this later on the course, in number theory. Instead of "divides", you could read $a \mid b$ as "b is a multiple of a" (or more precisely, "an integer multiple"), or "a is a factor of b".

Solution. Recall: a relation $\preceq$ on A is an *order relation* if it is reflexive, anti-symmetric, and transitive. - from lecture note

For any positive integer a, $a \mid a$ holds because $a = 1a$
$\Rightarrow \mid$ is reflexive

For any positive integer a and b, if $a \mid b$, and $b \mid a$ then $a = b$ as:
$a \mid b \Rightarrow b = ka, k \in \mathbb{Z}$
$b \mid a \Rightarrow a = mb, m \in \mathbb{Z}$
$\Rightarrow a = mka$
$\Rightarrow m = k = 1$ and $a = b, a, b \neq 0$ (positive integer)
$\Rightarrow \mid$ is anti-symmetric

For any positive integer a, b, c, if $a \mid b$ and $b \mid c$, then $a \mid c$:
$a \mid b \Rightarrow b = ka, k \in \mathbb{Z}$
$b \mid c \Rightarrow c = hb, h \in \mathbb{Z}$
$\Rightarrow c = (hk)a$
$\Rightarrow a \mid c$
$\Rightarrow \mid$ is transitive

Hence, $\mid$ is an order. A total order requires that for any positive integer $a, b, a \mid b$ or $b \mid a$ must hold. However, this is not the case, as illustrated with the counter-example: $2 \nmid 5, 5 \nmid 2$. Hence, $\mid$ is not a total order on positive integers.

3A5 (Same set, different orders) We define two different orders on $\mathbb{Z}^2$, the *pointwise order*

$$(a_1, a_2) \preceq_P (b_1, b_2) \quad \text{if and only if} \quad a_1 \leq b_1 \ \wedge \ a_2 \leq b_2$$

and the *lexical order*

$$(a_1, a_2) \preceq_L (b_1, b_2) \quad \text{if and only if} \quad (a_1 < b_1) \vee (a_1 = b_1 \ \wedge \ a_2 \leq b_2).$$

(a) Prove that both are orders.

(b) Which one(s) of them are total orders? Prove it.

(c) Prove that for all points $a, b \in \mathbb{Z}^2$, $a \preceq_P b$ implies $a \preceq_L b$.

(d) Visualize both orders by taking some point, say $b = (4, 2)$, and drawing the sets $\{a \in \mathbb{Z}^2 : a \preceq b\}$, where $\preceq$ is either $\preceq_P$ or $\preceq_L$.

(e) Give an example of a third order on $\mathbb{Z}^2$ (different from both $\preceq_P$ and $\preceq_L$) and visualize it.

Solution.

(a) Proof for *pointwise order*:

For any $(a_1, a_2) \in \mathbb{Z}^2, a_1 \leq a_1, a_2 \leq a_2 \Rightarrow (a_1, a_2) \preceq_P (a_1, a_2)$
$\Rightarrow \preceq_P$ is reflexive

If $(a_1, a_2) \preceq_P (b_1, b_2)$ and $(b_1, b_2) \preceq_P (a_1, a_2)$
$\Rightarrow a_1 \leq b_1 \leq a_1$ and $a_2 \leq b_2 \leq a_2$
$\Rightarrow (a_1, a_2) = (b_1, b_2)$
$\Rightarrow \preceq_P$ is anti-symmetric

If $(a_1, a_2) \preceq_P (b_1, b_2)$ and $(b_1, b_2) \preceq_P (c_1, c_2)$
$\Rightarrow a_1 \leq b_1 \leq c_1$ and $a_2 \leq b_2 \leq c_2$
$\Rightarrow a_1 \leq c_1$ and $a_2 \leq c_2$
$\Rightarrow (a_1, a_2) \preceq_P (c_1, c_2)$
$\Rightarrow \preceq_P$ is transitive

Hence, $\preceq_P$ is order.

Proof for *lexical order*:

For any $(a_1, a_2) \in \mathbb{Z}^2, a_1 < a_1$ is false, but $a_1 = a_1$ and $a_2 \leq a_2$ hold
$(a_1, a_2) \preceq_L (a_1, a_2)$
$\Rightarrow \preceq_L$ is reflexive

If $(a_1, a_2) \preceq_L (b_1, b_2)$, and $(b_1, b_2) \preceq_L (a_1, a_2)$
$\Rightarrow a_1 < b_1 < a_1$ is false and it must be that $a_1 = b_1$
Since $a_2 \leq b_2 \leq a_2$, $(a_1, a_2) = (b_1, b_2)$
$\Rightarrow \preceq_L$ is anti-symmetric

If $(a_1, a_2) \preceq_L (b_1, b_2)$ and $(b_1, b_2) \preceq_L (c_1, c_2)$
$\Rightarrow$ either $a_1 < b_1 < c_1$ holds or $a_1 = b_1 = c_1$ holds.
Since $a_2 \leq b_2 \leq c_2$ holds anyway, we have $(a_1, a_2) \preceq_L (c_1, c_2)$
$\Rightarrow \preceq_L$ is transitive

Hence $\preceq_L$ is order.

(b) A total order in this case is when either $(a_1, a_2) \preceq (b_1, b_2)$ or $(b_1, b_2) \preceq (a_1, a_2)$.

First we look at *pointwise order*:

Counter-example: (1,2) and (2,1)
Neither $(1, 2) \preceq_P (2, 1)$ nor $(2, 1) \preceq_P (1, 2)$ holds.
$\Rightarrow \preceq_P$ is not a total order.

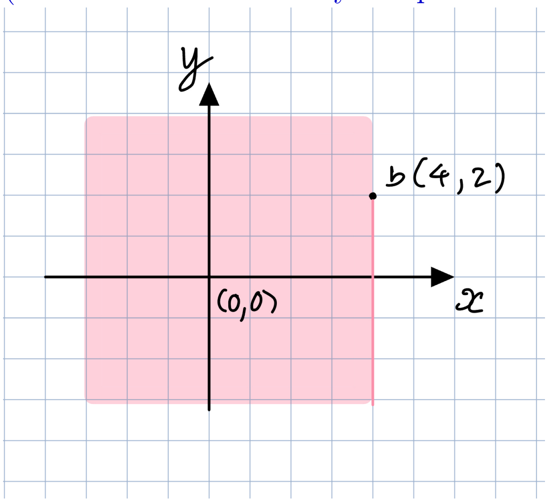
Secondly, let's look at *lexical order*:

For any $a, b \in \mathbb{Z}^2$, we have 2 cases:

- $a_1 \neq b_1$:
  Either $(a_1, a_2) \preceq_L (b_1, b_2)$ and $(b_1, b_2) \preceq_L (a_1, a_2)$ must hold, as it is true that either $a_1 < b_1$ or $b_1 < a_1$
- $a_1 = b_1$:
  It is either that $a_2 \leq b_2$, in which case $(a_1, a_2) \preceq_L (b_1, b_2)$ holds;
  Or $b_2 < a_2$, in which case $(b_1, b_2) \preceq_L (a_1, a_2)$ holds.
  Hence, $\preceq_L$ is a total order

(c) $a \preceq_P b$
$\Rightarrow a_1 \leq b_1 \wedge a_2 \leq b_2$
$\Rightarrow (a_1 = b_1 \vee a_1 < b_1) \wedge a_2 \leq b_2$
$\Rightarrow (a_1 = b_1 \wedge a_2 \leq b_2) \vee (a_1 < b_1 \wedge a_2 \leq b_2)$
In the both cases above, we can conclude that $a \preceq_L b$.
This means that $a \preceq_P b$ implies $a \preceq_L b$.

(d) With $b = (4, 2)$, for the set $\{a \in \mathbb{Z}^2 \ : \ a \preceq_P b\}$, the elements must satisfy $(a_1 \leq 4) \wedge (a_2 \leq 2)$. For the set $\{a \in \mathbb{Z}^2 \ : \ a \preceq_L b\}$, the elements must satisfy $(a_1 < 4) \wedge (a_1 = 4 \vee a_2 \leq 2)$.
Thus, for $\{a \in \mathbb{Z}^2 \ : \ a \preceq b\}$ where $\preceq$ is either $\preceq_P$ or $\preceq_L$, the elements must satisfy $(a_1 \leq 4) \wedge a_2 \leq 2) \vee (a_1 < 4)$.
We have the visualisation:
(Note that the set is only comprised of elements in $\mathbb{Z}^2$.)



(e) There are many valid solutions. The relation just needs to fulfill the three requirements (reflexivity, antisymmetry, transitivity). Reflexivity is probably easy to achieve. Antisymmetry, however, may cause a surprise. Note that any two *different* points a, b could be in the relation in *one direction*, that is $a \preceq b$ or $b \preceq a$, but *not both*. An order is not allowed to contain "loops", so that from a point you could move to a different "larger" point and then again "larger" and come back to the original point. The third allowed possibility is that they are not in relation in either way (two points can be *incomparable*).

- A simple example is the equality of both coordinates, $a_1 = b_1 \wedge a_2 = b_2$. Every point is comparable only to itself. This is a valid partial order (if not a very useful one).
- Another example is a linear order *within* each horizontal line, and incomparability of points on different lines: $a_1 \leq b_1 \wedge a_2 = b_2$. This illustrates that in a partial order, not all points need to be comparable with each other.
- A different kind of example comes from taking any already known order relation (e.g. pointwise or lexical order) and *reversing it*. E.g. the reverse of the pointwise order: $a_1 \geq b_1 \wedge a_2 \geq b_2$.

There are many other valid solutions.

3A6 (Manhattan) We define the *Manhattan distance* between points in $\mathbb{Z}^2$ as

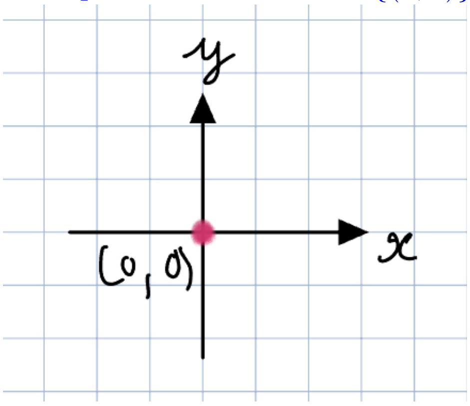
$$d((a_1, a_2), (b_1, b_2)) = |a_1 - b_1| + |a_2 - b_2|.$$

(You can think of a point as an intersection of streets in a city following a grid plan, and the distance is measured along the streets, with one block counting as one unit in either horizontal or vertical direction). We then define an equivalence relation R in $\mathbb{Z}$ by saying that $R(a, b)$ if and only if $d((0, 0), a) = d((0, 0), b)$. What are the equivalence classes? Visualize.

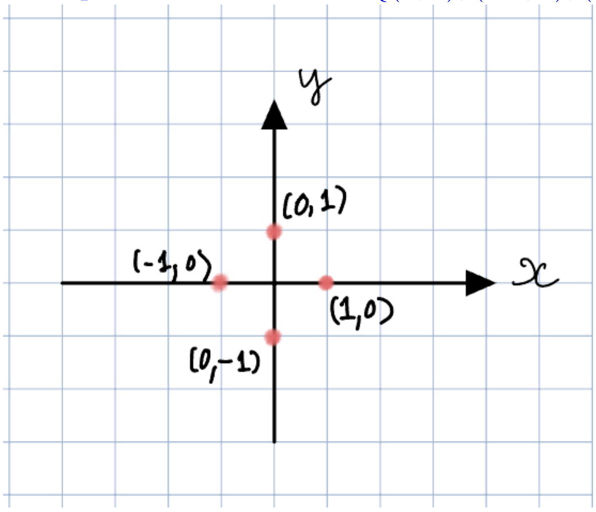
We usually think of square-shaped city blocks here. In reality the typical blocks on Manhattan are not squares at all, but rectangles whose "width" is much bigger than their "height".

Solution. Relation $R : R(a, b)$ if and only if $d((0, 0), a) = d((0, 0), b)$
This means that for each equivalence class, the points share the same Manhattan distance from the origin. Let's name C_k the set of those points that have distance k.

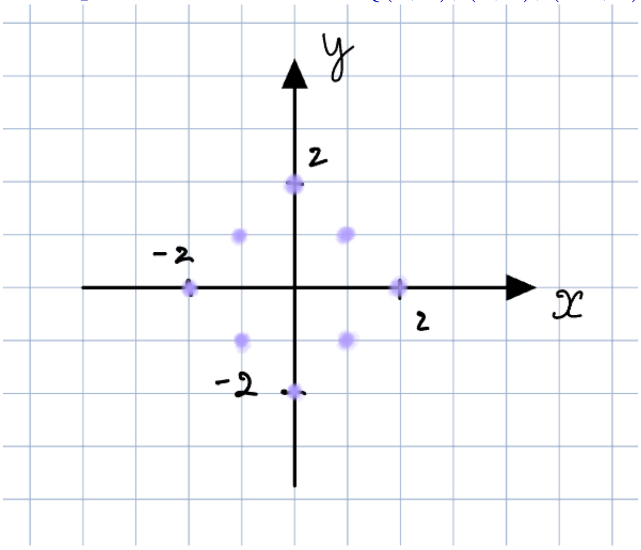
Equivalence class $C_0 = \{(0, 0)\}$:



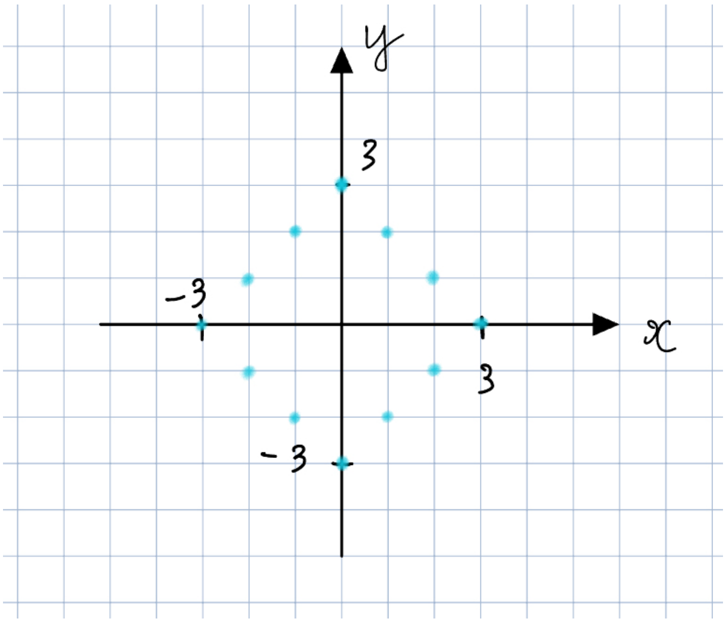
Equivalence class $C_1 = \{(1,0), (-1,0), (0,1), (0,-1)\}$:



Equivalence class $C_2 = \{(2,0), (0,2), (-2,0), (0,-2), (-1,1), (-1,-1), (1,1), (1,-1)\}$:



Equivalence class C_3:



and so on...