3B Relations, functions, cardinality

Some terminology: "Injection", "injective function" and "one-to-one function" are synonyms. ("Injection" is a noun, "injective" is an adjective.)

"Surjection", "surjective function" and "onto function" are synonyms. A surjection $f : A \to B$ is said to be "onto B" because it "covers" all of B.

"Bijection", "bijective function" and "one-to-one and onto function" are synonyms.

3B1 (Equivalences and orders) Let $\Omega = \{1, 2, 3\}$. We define two relations over subsets of Ω: let $E(S, T)$ be the relation $|S| = |T|$, and let $R(S, T)$ be the relation $|S| \leq |T|$. Recall that for finite sets, $|\ldots|$ means the number of elements.

Describe the two relations in words. Is E an equivalence relation? If yes, what are the equivalence classes? Is R an order relation? If yes, is it a total order?

Solution.

- Relation $E(S, T)$ holds when the sets S, T have the same number of elements (the same cardinality).
- Relation $R(S, T)$ holds when the number of elements for the set S is less than or equal to that of T.
- Let's take a closer look at E:
  For any set $A, E(A, A)$ must hold, sincbe any set has the same cardinality as itself.
  $\Rightarrow E$ is reflexive
  For any set A, B, if $E(A, B)$ holds, it means A, B have the same cardinality, and thus $E(B, A)$ must also hold.
  $\Rightarrow E$ is symmetric.
  For any set A, B, C, if $E(A, B)$ and $E(B, C)$ hold, it means that A, B have the same cardinality, and B, C have the same cardinality. $\Rightarrow A, C$ have the same cardinality. $\Rightarrow E(A, C)$ holds.
  $\Rightarrow E$ is transitive
  $\Rightarrow$ Thus, E is an equivalence relation.

  We have $\Omega = \{1, 2, 3\}$ so we have the ewuivalence classes $C_n, n \in N$ where the numerical value of n indicates the cardinality of the sets in each class:
  $C_0 = \{\varnothing\}$
  $C_1 = \{\{1\}, \{2\}, \{3\}\}$
  $C_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$
  $C_3 = \{\{1, 2, 3\}\}$

- Let's take a closer look at R:
  For any set $A, R(A, A)$ must hold, since any set has the same cardinality as itself. $\Rightarrow R$ is reflexive
  For any set A, B, if $R(A, B)$ and $R(B, A)$ both hold, it means that A, B have the same cardinality. However, this does not necessarily mean that they are the same set. For example, with $A = 1, 2$ and $B = 2, 3$, both $R(B, A), R(A, B)$ hold, yet $A \neq B$
  $\Rightarrow R$ is not anti-symmetric
  $\Rightarrow R$ is not an order

3B2 (Dual orders) If R is a relation on set A, its *dual* (or opposite, or converse) relation R^d is the relation defined by $R^d(x, y) \leftrightarrow R(y, x)$ for all $x, y \in A$ (we are viewing relations as predicates here). When using infix notation, a typical convention is to reverse the symbol, for example $\leq^d$ is written $\geq$, and $\subseteq^d$ is written $\supseteq$.

(a) Prove: If R is an order relation, then R^d is an order relation.

(b) Prove: If R is a total order, then R^d is a total order.

(c) If R is an order relation, what kind of relation is $R \wedge R^d$?

(d) What does the result from (c) say specifically for the relations $\leq$ (on real numbers), $\subseteq$ (on subsets of some set), and divisibility $\mid$ (on positive integers)?

Solution.

(a) If R is an order relation, R is reflexive, anti-symmetric, and transitive.

- R is reflexive $\Rightarrow$ for any $x \in A, R(x, x)$ holds $\Rightarrow R^d(x, y)$ holds $\Rightarrow R^d$ is reflexive
- R is anti-symmetric $\Rightarrow$ for any $x, y \in A$, if $R(x, y)$ and $R(y, x)$ both hold, then $x = y$
  Since we have $R^d(y, x) \leftrightarrow R(x, y)$, if both $R^d(x, y), R^d(y, x)$ hold, it must be that $x - y \Rightarrow R^d$ is anti-symmetric
- R is transitive $\Rightarrow$ For any $x, y, z \in A$, if $R(x, y), R(y, z)$ hold, $R(x, z)$ holds as well. We have:
  $R(x, y) \leftrightarrow R^d(y, x)$
  $R(y, z) \leftrightarrow R^d(z, y)$
  $R(x, z) \leftrightarrow R^d(z, x)$
  $\Rightarrow$ When $R^d(x, y), R^d(z, y)$ hold, $R^d(z, x)$ holds $\Rightarrow R^d$ is transitive

  Hence, R^d is an order if R is an order

(b) If R is a total order, for any $x, y \in A$, either $R(x, y)$ or $R(y, x)$ holds. By the given definition for R^d, $R^d(y, x) \leftrightarrow R(x, y)$ and $R^d(x, y) \leftrightarrow R(y, x)$
$\Rightarrow$ either $R^d(x, y)$ or $R^(y, x)$ must be true for any $x, y \in A$
$\mathbb{R}^d$ is a total order if R is a total order.

(c) $R \wedge R^d$ represents the set of pairs for which both $R(x, y), R^d(x, y)(x, y \in A)$ hold simultaneously. We have the definition of R^d as: $R^d(x, y) \leftrightarrow R(y, x)$
$\Rightarrow$ In $R \wedge R^d$, both $R(y, x), R(x, y)$ hold. If R is an order relation, it means that R is anti-symmetric $\Rightarrow$ meaning that for any $x, y \in A$, if both $R(y, x), R(x, y)$ hold, then $x = y$.
$\Rightarrow$ $R \wedge R^d$ only consists of pairs (x, y) where $x = y$ in the set A.
$\Rightarrow$ $R \wedge R^d$ is the identity relation.

(d)
- $\leq$:
  If R is the $\leq$ relation on real numbers, with the result in part (c), the relation $R \wedge R^d$ being the identity relation means that if $x \leq y$ and also $y \leq x, x = y$.
- $\subseteq$:
  Similarly, this means the 2 sets X, Y are the same set if $X \subseteq Y$ and $X \supseteq Y$ simultaneously.
- $|$:
  Similarly, this also means the 2 positive integers x, y are the same number if $x \mid y$ and $y \mid x$ simultaneously.

3B3 (Finite domains) Let $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$.

(a) How many different injections are there from A to B? (Hint: You can list them, but you could also just count. What are the possibilities for the value of $f(1)$? If you choose that $f(1)$ has a particular value, how many possibilities are then for the value of $f(2)$?)

(b) How many different surjections are there from A to B?

(c) How many different injections are there from B to A?

(d) How many different surjections are there from B to A?

(e) How many different bijections are there from A to A?

Solution.

(a) Injection from A to B means that every element in B is the image of *at most* 1 element in A.
$\Rightarrow$ No more than 1 element in A should be mapped to any element in B.
$\Rightarrow$ There are $4 \times 3 = 12$ different injections.

(b) Surjections from A to B means that for every element in B, there exist *at least* 1 element in A that the element in B is the image of.
$\Rightarrow$ There is no surjections from A to B.

(c) 0 injections from B to A.

(d) There are 4^2 functions from B to A. There are $4^2 - 2 = 14$ surjections from B to A, where we get the number of surjections by deducting the number of cases where the function is not a surjection from the total number of function. One case is when all 4 values in B are mapped to the value 1 in A, while no values in B map to the value 2 in A. The second case is when all 4 values in B are mapped to the value 2 in A, while no values in B map to the value 1 in A.

(e) 2 bijections.

3B4 (Infinite domains) Give an example of a function $\mathbb{N} \to \mathbb{N}$ that is

(a) injective but not surjective,

(b) surjective but not injective,

(c) bijective (that is, injective and surjective), *other than* the identity function $f(x) = x$,

(d) neither injective nor surjective.

Hint: Very simple functions should suffice in all cases. You don't need to construct very complicated ones. Try, for example, simple arithmetic (e.g. adding or subtracting a constant, or multiplying), or a constant function, or a function defined by cases. But make sure that your function is a well-defined function from $\mathbb{N}$ to $\mathbb{N}$, that is, for every $x \in \mathbb{N}$ you have $f(x) \in \mathbb{N}$.

Solution. Below are some examples:

(a) Some examples: $f(x) = 2x$, or $f(x) = x + 1$.

(b) Some examples: $g(x) = \lfloor \frac{x}{2} \rfloor$ (where $\lfloor \ldots \rfloor$ is the floor function, or "rounding down"); or $g(0) = 0$ and $g(x) = x - 1$ for $x > 0$.

(c) For example, $h(x) = \begin{cases} x-1, & \text{if } x \text{ is odd} \\ x+1, & \text{if } x \text{ is even} \end{cases}$
Note: on this course we have defined $\mathbb{N}$ to include 0.

(d) Some examples: $k(x) = x \mod 2$, or a constant function like $f(x) = 7$.

3B5 (Function composition) Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove: If f and g are injections, then $g \circ f$ is an injection.

(b) Prove: If f and g are surjections, then $g \circ f$ is a surjection.

(c) Prove: If f and g are bijections, then $g \circ f$ is an bijection.

Hint: For injection, it is perhaps easiest to work with the contrapositive form: f is injective if $x \neq y \Rightarrow f(x) \neq f(y)$.

Part (c) shows that our definition of "having same cardinality", which is based on the existence of a bijection, is a *transitive* relation between sets, even for infinite sets: if A, B have the same cardinality, and B, C have the same cardinality, then so do A, C.

Solution.

(a) From lecture note: $\forall x, y \in A : f(x) = f(y) \to x = y$
Contrapositive: $\forall x, y \in A : x \neq y \to f(x) \neq f(y)$
If $x \neq y$, then $f(x) \neq f(y)$ (because f is an injection). Then because $f(x) \neq f(y)$, also $g(f(x)) \neq g(f(y))$ (because g is an injection). In other words $(g \circ f)(x) \neq (g \circ f)(y)$. This proves that $(g \circ f)$ is an injection.

(b) For every $c \in C$, we have to find an element $a \in A$ such that $g(f(a)) = c$. First, given c, we can find an element $b \in B$ such that $g(b) = c$, because g is a surjection. Then we can find an element $a \in A$ such that $f(a) = b$, because a is a surjection. Putting these together, we note that $g(f(a)) = g(b) = c$ as required.

(c) As bijection is both injection and surjection, based on (a) and (b), (c) follows.

3B6 (Cardinality of power sets) Prove, by explicitly constructing an injection, that if A is any set (finite or infinite), then $|A| \leq |P(A)|$, where $P(A)$ means the power set. Hint: There is a very simple injection.

For finite sets we know that if $|A| = n$, then $|P(A)| = 2^n$, so this exercise proves, in a set-theoretic way, that $2^n \geq n$ for all $n \in \mathbb{N}$. Of course this inequality could be proved in other ways, for example by induction.

Solution. An example of an injection is $a \mapsto \{a\}$, in other words $f(a) = \{a\}$. This function f is an injection, since it is defined for all $a \in A$, and distinct elements in A are mapped to distinct sets. If $f(a) = f(b)$, then $\{a\} = \{b\}$, implying that $a = b$.

Because there is an injection from A to $P(A)$, we have $|A| \leq |P(A)|$.

3B7 (Cardinality of intervals) Recall that $[a, b]$ and $]a, b[$ denote closed and open intervals of real numbers. We say that two sets A and B are *equipotent*, or "have the same cardinality", if there exists a bijection between them.

(a) Prove, by constructing a bijection, that every nonempty closed interval $[a, b]$ (where $a < b$) is equipotent to the closed unit interval $[0, 1]$. Hint: It is enough to construct a bijection between the two sets in one direction. Which direction is easier?

(b) Prove, by constructing a bijection, that every nonempty open interval $]a, b[$ (where $a < b$) is equipotent to the open unit interval $]0, 1[$.

(c) From the previous parts, conclude that *all* nonempty closed intervals have the same cardinality, and *all* nonempty open intervals have the same cardinality.

(d) Construct an injection from $]0, 1[$ to $[0, 1]$.

(e) Construct an injection from $[0, 1]$ to $]0, 1[$.

It is a general fact (but not quite trivial to prove) that if there are injections both ways $f : A \to B$ and $g : B \to A$, then there is also a bijection $h : A \to B$. This is known as the Schröder–Bernstein theorem. We will not prove this on the course (but look up the proof if you are interested). From (d) and (e), and applying the Schröder–Bernstein theorem, we can conclude that the closed interval $[0, 1]$ and the open interval $]0, 1[$ are equipotent.

Solution.

(a) A simple bijection from $[0, 1]$ to $[a, b]$ is $f(x) = a+(b-a)x$. Clearly, if $0 \leq x \leq 1$, then $a \leq f(x) \leq b$, so the function is well-defined (the values are in the codomain). Also, if $f(x) = f(y)$, then $a+(b-a)x = a+(b-a)y$, from which it follows (by algebra) that $x = y$; so f is an injection. Finally, for any $u \in [a, b]$, we can choose $x = \frac{u-a}{b-a} \in [0, 1]$ such that $f(x) = u$, so f is a surjection.

(b) We can again use $f(x) = a + (b - a)x$. The same proofs work as in (a), by the straightforward changes to exclude the endpoints (both in the domain and in the codomain).

(c) Every bijection f has an inverse function f^{-1} which is also a bijection. Suppose we have two closed intervals $[a, b]$ and $[c, d]$. By (a), there exist bijections $f_1 : [0, 1] \to [a, b]$ and $f_2 : [0, 1] \to [c, d]$. Then $f_2 \circ f_1^{-1}$ is a bijection from $[a, b]$ to $[c, d]$, so $[a, b]$ and $[c, d]$ have the same cardinality.

Between any two open intervals $]a, b[$ and $]c, d[$ we can similarly construct a bijection, using the bijections from (b).

(d) $f(x) = x$

(e) $f(x) = \frac{1}{4} + \frac{x}{2}$.

The challenge problem is worth an extra point.

3B8 (** CHALLENGE: Cardinality of powersets, part II) Prove that if A is a set (either finite or infinite), there is no bijection from A to its powerset $P(A)$.

Hint: It is enough to show that there is no surjection. Suppose that f is a function from A to $P(A)$. For each element $x \in A$, either $x \in f(x)$ or not. Now consider the set

$$T = \{x \in A \;:\; x \notin f(x)\}.$$

Prove that T is not equal to $f(y)$ for any $y \in A$. That means that f is not a surjection to $P(A)$, because there is an element $T \in P(A)$ not covered by f.

Together with 3B6, this exercise shows that for any set A, the powerset $P(A)$ has different (bigger) cardinality than A. For example, because we can form power sets of power sets,

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < \dots$$

In particular, there are *infinitely many* cardinalities that infinite sets can have, not just two ("same as $\mathbb{N}$" and "same as $\mathbb{R}$").

Solution. Suppose that f is a function from A to $P(A)$. So, for every $x \in A$, there exists an image $f(x)$ which is a subset of A.

Following the hint in the problem statement we define

$$T = \{x \in A \;:\; x \notin f(x)\}.$$

That is, T consists of *those* elements x that are *not* elements of their own images (under f). *For example*, if $A = \mathbb{N}$ and $f(5) = \{3, 6, 9\}$, then $5 \in T$ because 5 is not an element of its own image. But if $f(5) = \{5\}$, or if $f(5) = \{3, 5, 10, 37\}$, then $5 \notin T$, by our definition of the set T.

Now we prove that T is not the image of *any* element $y \in A$. Suppose, in order to prove a contradiction, that there is an element y such that $f(y) = T$. Consider two cases.

Case 1: $y \in f(y)$. Then by definition $y \notin T$. But $T = f(y)$, so $y \notin f(y)$, a contradiction.

Case 2: $y \notin f(y)$. Then by definition $y \in T$. But $T = f(y)$, so $y \in f(y)$, again a contradiction.

Because both cases are impossible, we see that T does not equal $f(y)$ for any $y \in A$. So f is not a surjection (hence not a bijection).

We have proven that *whatever function* $f : A \to P(A)$ we consider, f is not a bijection. In other words, there *does not exist* a bijection from A to $P(A)$. $\square$