# 4B Enumerative combinatorics II

**4B1** (Higher derivatives) Recall that the (first) derivative of $x^n$ (with respect to $x$) is $nx^{n-1}$, and the $(k+1)$th derivative is the derivative of the $k$th derivative.

(a) What are the second, 3rd, 4th, 5th and 6th derivatives of $x^5$?

(b) Generally, if $1 \leq k < n$, what is a simple expression for the $k$th derivative of $x^n$, expressed using the falling product?

(c) What is a simple expression for the $n$th derivative of $x^n$, expressed using the factorial?

(d) What is the $k$th derivative of $x^n$ when $k > n$?

**Solution.**

(a) First: $5x^4$, Second: $20x^3$, Third: $60x^2$, Fourth: $120x$, Fifth: $120$, Sixth: $0$

(b)
$$n(n-1)...(n-k+1)x^{n-k} = n^{\underline{k}}x^{n-k}$$

(c)
$$n(n-1)...(n-n+1)x^{n-n} = n!x^0 = n!$$

(d) When k > n, the derivative is taken from a constant (see part c). The derivative of a constant is always zero.

It may be useful to observe when $k > n$, the falling product $n^{\underline{k}}$ is actually zero. Indeed, already at $k = n+1$ we have

$$n^{\underline{n+1}} = \underbrace{n(n-1)(n-2)\dots(2)(1)(0)}_{n+1 \text{ factors}} = 0,$$

because the $(n+1)$th factor is zero. This is compatible with our "consecutive choice" interpretation. Out of a set of $n$ elements, there is *no way* (*zero ways*) to choose $n+1$ different objects. For example, $4^{\underline{5}} = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 = 0$.

**4B2** (Inclusion-exclusion) A DNA sequence is a string (tuple) of letters, each chosen from the set $\{A, C, G, T\}$. For example, $TAGGA$ is a possible sequence of length 5.

(a) How many DNA-sequences of length 5 exist that contain each of the letters $A, C, G$ at least once?

(b) (OPTIONAL, not required for scoring the problem, and no extra points.) How many DNA-sequences of length 5 exist that contain each of the letters $A, C, G, T$ at least once?

**Solution.**

(a) One way to solve this is to calculate the number of strings not containing A, C and G at least once and subtract this from the total number of possible strings.

The total number of strings is calculated using the multiplication principle: $4^5 = 1024$.

The number of strings not containing A, C and G at least once can be calculated using the IE-principle. Denote the sets of strings NOT containing A, C or G as $S_A$, $S_C$ and $S_G$ respectively. Since we can pick from any of the 3 leftover letters for each, all of their cardinalities are $|S_A| = |S_G| = |S_C| = 3^5 = 243$.

For $|S_A \cap S_C|$, we have two letter leftover to choose from. This applies to all the other 2-element intersections, so $|S_A \cap S_C| = |S_C \cap S_G| = |S_A \cap S_G| = 2^5 = 32$.

Finally, for $|S_A \cap S_C \cap S_G|$ we only have one letter (namely T) left to choose from, so $|S_A \cap S_C \cap S_G| = 1^5 = 1$. (That one string is TTTTT.)

Now using the IE-principle, we have

$$\begin{aligned}
&|S_A \cup S_C \cup S_G| \\
&= |S_A| + |S_G| + |S_C| - |S_A \cap S_C| - |S_C \cap S_G| - |S_A \cap S_G| + |S_A \cap S_C \cap S_G| \\
&= 3 \cdot 243 - 3 \cdot 32 + 1 \\
&= 634.
\end{aligned}$$

And finally, subtracting this number from the total number of strings we get $1024 - 634 = \mathbf{390}$.

(b) As in (a) we use $S_k$ to denote the set of strings that do not contain the letter $k$. All of the single sets $S_A, S_C, S_G, S_T$ have cardinality 243 as in (a), but there are now four of them. All intersections of two and three sets have cardinalities 32 and 1 as in (a), but there are more of them: there are *six* pairwise intersections, because from four letters you can choose $\binom{4}{2} = 6$ unordered pairs. And four triple-intersections because from four sets you can choose $\binom{4}{3} = 4$ triples.

Furthermore, in the inclusion-exclusion equality, we have one term for the four-way intersection $S_A \cap S_C \cap S_G \cap S_T$. Which happens to be empty, because there is no way to create a string of five letters if all letters are forbidden. So the cardinality of this intersection is zero. (This is *not* a general fact of 4-way inclusion-exclusion; but *here* that set is empty.)

So we have

$$
\begin{aligned}
&|S_A \cup S_C \cup S_G \cup S_T| \\
=&|S_A| + |S_G| + |S_C| + |S_T| && \text{(4 terms)} \\
&- |S_A \cap S_C| - \ldots - |S_G \cap S_T| && \text{(6 terms)} \\
&+ |S_A \cap S_C \cap S_G| + \ldots + |S_C \cap S_G \cap S_T| && \text{(4 terms)} \\
&- |S_A \cap S_C \cap S_G \cap S_T| && \text{(1 term)} \\
=&4 \cdot 243 - 6 \cdot 32 + 4 \cdot 1 - 1 \cdot 0 \\
=&784.
\end{aligned}
$$

That was the strings which are lacking at least one letter. The complement, strings that contain all four letters, have the cardinality $4^5 - 784 = 1024 - 784 =$ **240**.

For reference, here are the strings, listed in alphabetical order. Do you agree that (1) there are 240 of them, (2) each of them contains all four letters, (3) every such string has been listed, and (4) every such string has been listed only once?

Verifying that there are no duplicates in the list could be very laborious if the list was in arbitrary order — if we to compare every string against every other string, it is $\binom{240}{2} = 28680$ comparisons. Now that they are in alphabetical order, it would enough to check every pair of *consecutive* strings in the list.

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AACGT AACTG AAGCT AAGTC AATCG AATGC ACAGT ACATG ACCGT ACCTG ACGAT ACGCT ACGGT ACGTA ACGTC ACGTG
ACGTT ACTAG ACTCG ACTGA ACTGC ACTGG ACTGT ACTTG AGACT AGATC AGCAT AGCCT AGCGT AGCTA AGCTC AGCTG
AGCTT AGGCT AGGTC AGTAC AGTCA AGTCC AGTCG AGTCT AGTGC AGTTC ATACG ATAGC ATCAG ATCCG ATCGA ATCGC
ATCGG ATCGT ATCTG ATGAC ATGCA ATGCC ATGCG ATGCT ATGGC ATGTC ATTCG ATTGC CAAGT CAATG CACGT CACTG
CAGAT CAGCT CAGGT CAGTA CAGTC CAGTG CAGTT CATAG CATCG CATGA CATGC CATGG CATGT CATTG CCAGT CCATG
CCGAT CCGTA CCTAG CCTGA CGAAT CGACT CGAGT CGATA CGATC CGATG CGATT CGCAT CGCTA CGGAT CGGTA CGTAA
CGTAC CGTAG CGTAT CGTCA CGTGA CGTTA CTAAG CTACG CTAGA CTAGC CTAGG CTAGT CTATG CTCAG CTCGA CTGAA
CTGAC CTGAG CTGAT CTGCA CTGGA CTGTA CTTAG CTTGA GAACT GAATC GACAT GACCT GACGT GACTA GACTC GACTG
GACTT GAGCT GAGTC GATAC GATCA GATCC GATCG GATCT GATGC GATTC GCAAT GCACT GCAGT GCATA GCATC GCATG
GCATT GCCAT GCCTA GCGAT GCGTA GCTAA GCTAC GCTAG GCTAT GCTCA GCTGA GCTTA GGACT GGATC GGCAT GGCTA
GGTAC GGTCA GTAAC GTACA GTACC GTACG GTACT GTAGC GTATC GTCAA GTCAC GTCAG GTCAT GTCCA GTCGA GTCTA
GTGAC GTGCA GTTAC GTTCA TAACG TAAGC TACAG TACCG TACGA TACGC TACGG TACGT TACTG TAGAC TAGCA TAGCC
TAGCG TAGCT TAGGC TAGTC TATCG TATGC TCAAG TCACG TCAGA TCAGC TCAGG TCAGT TCATG TCCAG TCCGA TCGAA
TCGAC TCGAG TCGAT TCGCA TCGGA TCGTA TCTAG TCTGA TGAAC TGACA TGACC TGACG TGACT TGAGC TGATC TGCAA
TGCAC TGCAG TGCAT TGCCA TGCGA TGCTA TGGAC TGGCA TGTAC TGTCA TTACG TTAGC TTCAG TTCGA TTGAC TTGCA
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**4B3** (Round table) Six guests ABCDEF are to be seated around a round table. The host wants to list all possible arrangements (and then ponder which arrangement is the best for good discussions). What matters is who sits as the left neighbor of whom, and who sits as the right neighbor of whom. If an arrangement is rotated clockwise or counterclockwise, the host considers it the same arrangement (listed only once).

Guests A and B cannot stand each other, so arrangements that seat them as neighbors are not possible. Count the possible arrangements.

Hint: The number is almost a hundred, so it is probably best for *you* not to list all arrangements. But you could list some "partial" arrangements (make some choices concerning one or two guests), and then count the remaining choices by arithmetic.

**Solution.**

One way of solving this is to make partial arrangements about who sits next to A. A can sit with C and E, C and F, D and E... etc. We can choose 2 out of the 4 available guests (CDEF, not including B) in $\binom{4}{2} = 6$ ways.

Having chosen two guests to sit with A, the other 3 can be chosen to sit in $3 \cdot 2 \cdot 1 = 3! = 6$ ways. Additionally, the two sitting with A can be placed in two ways (one on the left side, one on the right side).

Collecting all choices,

- 6 ways to choose the two neighbors of A,
- 6 ways to arrange the remaining three guests,
- 2 ways to order the neighbors of A,

leading to $6 \cdot 6 \cdot 2 = 72$ ways.

There are **many** ways to arrive at the same result. For example: Start from the place where A is seated. "Without loss of generality", let's say A is seated on the south side of the table. (Any arrangement can be rotated so that this is true.) Then choose a place for B. Because of the restriction, there are 3 possible places for B (out of the six places, forbid the place where A already is, and A's two neighbors). The remaining four guests CDEF can be placed freely in the remaining four places. Collecting all choices:

- 3 ways to place B,
- $4! = 24$ ways to place CDEF,

for a total of $3 \cdot 24 = 72$.

**4B4** (Back to the lab) We are back in the metallurgic laboratory (cf. 4A1). Now we are considering mixtures of $k$ substances (we think of iron as just one possible substance). We measure the amount of each substance in *tenths* of the total mass, and we only consider integer numbers of the tenths. For each substance, any number from 0 to 10 is possible, but they must add up to ten: for example, if given three substances, some of their possible mixtures are $(5, 4, 1)$, $(10, 0, 0)$ and $(3, 3, 4)$.

(a) We are considering two substances $A$ and $B$. What are their possible mixtures, and how many are they?

(b) We are considering three substances $A$, $B$ and $C$. How many possible mixtures are there? (Hint: Consecutive choice. If you have chosen the amount of $A$ to be $x$ tenths, how many choices do you have for the amount of $B$?)

(c) We are considering $k$ substances $A_1, A_2, \ldots, A_k$. Find a general formula, involving binomial coefficients, for the number of mixtures these substances. (Hint: Stars and bars from lecture 7, or lecture notes §2.2.2)

(d) Apply your general formula to the cases $k = 1, 2, 3, 4$. Does the result for $k = 1$ make sense? Do the results for 2 and 3 match what you calculated in (a)–(b)?

**Solution.**

(a) The combinations are (0,10), (1,9), (2,8), (3,7), (4,6), (5,5) and five more the "other way" ((6,4) etc.), so 11 in total.

Or: the amount of the first substance is one of the integers from 0 to 10, that is 11 choices. Having chosen the first amount as $x$, the other is fully determined as $10 - x$, so that was all there was to choose.

(b) If we choose A $=$ 10, we have 1 choice for B. If we choose A $=$ 9, we have 2 choices for B. If we choose A $=$ 8, we have 3 choices for B. ... If we choose A $=$ 0, we have 11 choices for B.

So we get $1 + 2 + 3 + ... + 10 + 11 = 66$ possible choices and thus 66 possible mixtures.

The sum of consecutive integers from 1 to 11 can be calculated by just performing the ten additions (some amount of work!), or one can recognize that such a sum is a so-called triangular number (mentioned on lecture 6). Then it is easy: The sum from 1 to 11 is the 11th triangular number $T_{11} = 11 \cdot (11+1)/2 = 66$.

Note that we had to use the *addition principle*, because different initial choices (amounts of A) led to *different numbers* of available choices later (for B). We could not use the multiplication principle.

If the sum were longer, direct addition would be quite laborious *and* prone to accidental errors. There is a famous story of Gauss, as a schoolboy, doing the sum $1 + 2 + \ldots + 100$ quickly as $(100 \times 101)/2 = 5050$, while his classmates spent ages doing the direct additions. Moreover, Gauss's answer was the only correct one — easy to believe, because doing 99 additions without making a single error is not easy.

(c) We have k different substances, and we want them to be added in some tenths of total mass such that the total mass is 10. Using the lecture notes, we get $\binom{k+10-1}{10} = \binom{k+9}{10}$

(d)
- $k = 1$: $\binom{10}{10} = 1$. This makes sense because there is only one way we can combine the substance with itself.
- $k = 2$: $\binom{11}{10} = 11$, matching (a).
- $k = 3$: $\binom{12}{10} = 66$, matching (b).
- $k = 4$: $\binom{13}{10} = 286$

**4B5** (Discrete maximization) Consider the function $f : \{0, 1, 2, \ldots, n\} \to \mathbb{R}$,

$$f(k) = \binom{n}{k} p^k q^{n-k},$$

where $n = 50$, $p = 0.2$ and $q = 0.8$ are given constants. We want to find the integer $k$ where $f(k)$ attains its maximum. If $f$ were a continuously differentiable function from real numbers, we would probably begin by finding where its derivative is zero. But our $f$ is defined only on integer values of $k$. So we will do something analogous in our discrete situation.

(a) For an arbitrary $k \in \{0, 1, 2, \ldots, n-1\}$, express the *quotient*

$$Q(k) = \frac{f(k+1)}{f(k)}$$

in a simple form. Hint: In 4A4 you have already worked out the quotient $\binom{n}{k+1}/\binom{n}{k}$. Now you just have some extra factors, and many of them should cancel out. Observe that $Q(k) > 1$ means that $f(k+1) > f(k)$, and similarly for "=" and "<".

(b) Using your formula from (a), find when $Q(k) > 1$. You should be able to find a value $k^*$ such that for all $k < k^*$ we have $Q(k) > 1$, and for all $k \geq k^*$ we have $Q(k) < 1$.

Hint: You can solve the inequality $Q(k) > 1$ algebraically, rearranging it so that $k$ is on one side and everything else is on the other side. Remember that $n, p, q$ are constants. If it helps, you can begin by studying some numerical values of $Q(k)$ in the interval $7 \leq k \leq 13$, to get a feeling of how the function behaves.)

(c) Can you now say that $f(k^*)$ is the largest value that $f$ takes? Why?

(d) Calculate the three values $f(k^* - 1), f(k^*), f(k^* + 1)$, verifying that at least locally $f(k^*)$ is indeed a maximum, that is $f(k^* - 1) < f(k^*) > f(k^* + 1)$.

In this exercise we have learned a method for finding the maximum of a function whose domain is discrete. Finding $k^*$ is analogous to finding a zero of the derivative: it is the point where $f$ turns from increasing to decreasing.

The same method can be applied generally, with other functions. An alternative would be to study the *differences* $f(k+1) - f(k)$, but in combinatorics, quotients often have nicer form.

**For those who already know something about probabilities:** The function $f$ is the so-called *probability mass function* of a binomial distribution. Suppose we have a biased coin that has probability $p = 0.2$ for "heads", and probability $q = 0.8$ for "tails" whenever we toss it. If we toss it $n = 50$ times, then $f(k)$ is the probability that we obtain exactly $k$ "heads" (and $n-k$ "tails"). In this exercise we have found, by combinatorial calculations, the most probable number of heads ($k^*$) and its probability ($f(k^*)$).

**Solution.**

(a)

$$Q(k) = \frac{f(k+1)}{f(k)} = \frac{\binom{n}{k+1} \cdot p^{k+1} \cdot q^{n-k-1}}{\binom{n}{k} \cdot p^k \cdot q^{n-k}}$$

We know from Problem 4A4 that $\binom{n}{k+1}/\binom{n}{k} = \frac{n-k}{k+1}$. Now

$$Q(k) = \frac{(n-k) \cdot p^k \cdot p \cdot q^{n-k-1}}{(k+1) \cdot p^k \cdot q^{n-k-1} \cdot q} = \frac{(n-k)p}{(k+1)q}$$

(b) Solve the inequality $Q(k) > 1$ for $k$:

$$\begin{aligned}
& Q(k) > 1 \\
\iff \quad & \frac{(n-k)p}{(k+1)q} > 1 \\
\iff \quad & (n-k)p > (k+1)q \qquad \text{(mult. by pos. quantity)} \\
\iff \quad & np - kp > kq + q \\
\iff \quad & k < \frac{np - q}{p+q} = \frac{50 \cdot 0.2 - 0.8}{0.2 + 0.8} \\
\iff \quad & k < 9.2.
\end{aligned}$$

So $Q(k) > 1$ for all $k \leq 9$ (remember that $k$ is an integer). For all $k \geq 10$ we have $Q(k) < 1$. Note that equality $Q(k) = 1$ would require that $k = 9.2$, which is not possible. So we choose $k^* = 10$.

The double arrows $\iff$ between the inequalities are not decoration, or something that one can throw in without thought, to make a proof look "more mathematical", or to express that "this is the next step of something, although I don't know how it is related to the previous step."

They express the claim that two inequalities are *equivalent*, that is, if one of them is true, then the other is true. When posing such a claim, one should know that the implication works both ways. Since we have a chain of them, we know that the first and the last inequalities are equivalent (true for the same values of $k$, and false for the same values of $k$.)

Perhaps you have a custom of writing only one-directional implication arrows between consecutive steps. That is all right, if you only need to claim the implication in one direction. Be careful it is the direction that you want to claim.

(c) Yes. When $Q(k) > 1$, we know that $f(k+1) > f(k)$ so the function is increasing. When $Q(k) < 1$, we know that $f(k+1) < f(k)$ so the function is decreasing.

We have chosen a $k^* = 10$ so that:

- For all $k < k^* = 10$ (that is $k \leq 9$), we have $f(k+1) > f(k)$. In other words
$$f(0) < f(1) < \ldots < f(9) < f(10).$$
- For all $k \geq k^* = 10$ we have $f(k+1) < f(k)$. In other words
$$f(10) > f(11) > \ldots > f(49) > f(50).$$

Together these two chains of inequalities show that $f(k) < f(10)$ for every $k \neq 10$, so $f(10)$ is indeed the largest of all values that $f$ takes; a *global maximum* over the whole domain $\{0, 1, \ldots, 50\}$.

(d) $f(9) = \binom{50}{9} \cdot (0.2)^9 \cdot (0.8)^{41} \approx 0.1364$
$f(10) = \binom{50}{10} \cdot (0.2)^{10} \cdot (0.8)^{40} \approx 0.1398$
$f(11) = \binom{50}{11} \cdot (0.2)^{11} \cdot (0.8)^{39} \approx 0.1271$
Looking at these three values only, we see that $f(9) < f(10) > f(11)$, so the point $k = 10$ is at least a *local* maximum. That is, the neighboring values of $f$ are not bigger than $f(10)$.

Part (d) is a good *sanity check* for (c), because a point cannot be a global maximum without being a local maximum. If $f(k^*)$ is the biggest of *all* values of $f$, surely it must be the biggest of the three when you compare only with the two neighbors.

However, (d) does not *prove* (c), because some function *could* increase and decrease back and forth, many times, having many local maximums. In part (c) we actually proved more: we proved that *this* function $f$ does not do that. It steadily increases on every step from $f(0)$ to $f(10)$, and then steadily decreases on every step from $f(10)$ to $f(50)$.

Figure 1 shows a plot of $f$ and $Q$ near the maximum point. Notice the transit from $Q > 1$ to $Q < 1$ without ever visiting $Q = 1$, unlike continuous functions in calculus. (The circles are the function values — the connecting lines are just for visualization.) The picture says nothing about the functions outside of what is shown, so it is not a proof that $f(10)$ is the global maximum.

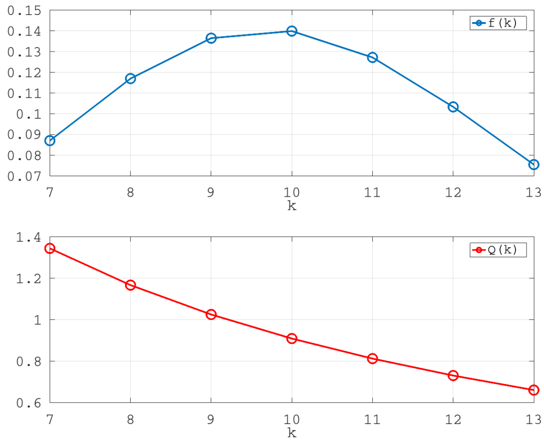


Figure 1: Functions $f$ and $Q$ (partial view)

**4B6** (Twelve days of Christmas) In a popular Christmas song the protagonist receives increasing numbers of gifts every day. Here are the first five days. In the song it continues in the same manner for twelve days.

On the first day of Christmas, my true love gave to me
A partridge in a pear tree.
On the second day of Christmas, my true love gave to me
Two turtle doves,
And a partridge in a pear tree.
On the third day of Christmas, my true love gave to me
Three French hens,
Two turtle doves,
And a partridge in a pear tree.
On the fourth day of Christmas, my true love gave to me
Four calling birds,
Three French hens,
Two turtle doves,
And a partridge in a pear tree.
On the fifth day of Christmas, my true love gave to me
Five golden rings,
Four calling birds,
Three French hens,
Two turtle doves,
And a partridge in a pear tree.

(a) How many gift items are given on days 1, 2, 3, 4 and 5? (Just add them up. E.g. three French hens are *three* items. A partridge is one item. Don't count the pear tree: it is not a gift, the partridge is just sitting there.)

(b) Generally, how many gift items are given on the $k$th day, when $k$ is any positive integer? Assume that the number of gifts keeps increasing in the same manner. Start by writing it as a big sum (using $\sum$), but then give a general formula using less than ten basic arithmetic operations (no big sum). Hint: Lecture 6.

(c) Using the numbers from (a), calculate the *running totals* (or *cumulative sums*) on days 1 to 5. That is, on each day, calculate how many items have been given so far (including that day). Just the numbers, no general formula required here. Note that each day a *new* partridge is given, and two *new* turtle doves etc.

(d) Go to `https://oeis.org/`, enter your numbers from (c), and hit Search. From the results, find how many items are given over the twelve days, in total. Search for "Christmas" on the result page to see if it is mentioned.

(e) (OPTIONAL – Not required for scoring the problem, and no extra points.) OEIS gives a simple general formula for the total number of items given during the first $n$ days. Try to prove, e.g. by induction, that the formula is correct.

**Solution.**

(a) Day 1: 1

Day 2: $2 + 1 = 3$

Day 3: $3 + 2 + 1 = 3 + 3 = 6$

Day 4: $4 + 3 + 2 + 1 = 4 + 6 = 10$

Day 5: $5 + 4 + 3 + 2 + 1 = 5 + 10 = 15$

(b)

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

This is again a sum of consecutive integers from 1 to $k$, that is, the triangular number $T_k$.

(c) Day 1: 1

Day 2: $1 + 3 = 4$

Day 3: $6 + 4 = 10$

Day 4: $10 + 10 = 20$

Day 5: $15 + 20 = 35$

(d) 364.

From the result page we see that this same sequence appears in many different contexts. Among other things, it is described as "the number of gifts received from the lyricist's true love up to and including day n in the song The Twelve Days of Christmas".

(e) Let $G(n)$ be the number of gifts up to day $n$, that is

$$G(n) = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)/2,$$

using our triangular number formula from (b).

The OEIS entry A000292 defines a function

$$a(n) = n(n+1)(n+2)/6. \qquad (*)$$

We want to prove that $G(n) = a(n)$ for all $n \geq 1$. By direct calculation we see that $G(n) = a(n)$ for $1 \leq n \leq 3$:

$$\begin{aligned} a(1) &= (1 \cdot 2 \cdot 3)/6 = 1 = G(1) \\ a(2) &= (2 \cdot 3 \cdot 4)/6 = 4 = G(2) \\ a(3) &= (3 \cdot 4 \cdot 5)/6 = 10 = G(3) \end{aligned}$$

This serves as the base case for induction. (Case $n = 1$ would be sufficient, but it does not hurt to check some more cases directly.)

As the induction hypothesis we assume that $n \geq 1$ and

$$G(n) = a(n).$$

For the induction step we claim that $G(n+1) = a(n+1)$. It turns out easiest to start from both sides of this claimed equality, and manipulate each one until we reach the same form. On the LHS we have

$$\begin{aligned} G(n+1) &= G(n) + T_{n+1} && \text{(separating last term)} \\ &= \frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} && \text{(by induction hypothesis)} \\ &= \frac{n^3 + 3n^2 + 2n}{6} + \frac{3n^2 + 9n + 3}{6} && \text{(expanding products)} \\ &= \frac{n^3 + 6n^2 + 11n + 3}{6}. && \text{(combining like powers)} \end{aligned}$$

On the RHS we have

$$\begin{aligned} a(n+1) &= \frac{(n+1)(n+2)(n+3)}{6} \\ &= \frac{n^3 + 6n^2 + 11n + 3}{6}. && \text{(expanding products)} \end{aligned}$$

Both expressions are the same, so we have proved $G(n+1) = a(n+1)$. □

Note that expanding a product into a longhand polynomial is usually easier than the other direction, converting a polynomial into a factored form. So after the row marked "combining like powers", instead of trying to factor it (towards what we have on the RHS), it was easier to start anew from the RHS quantity $a(n+1)$ and expand it.

This is very typical in induction proofs. When trying to prove that some LHS equals some RHS, it may be a good idea to start from both ends and try to make them meet.