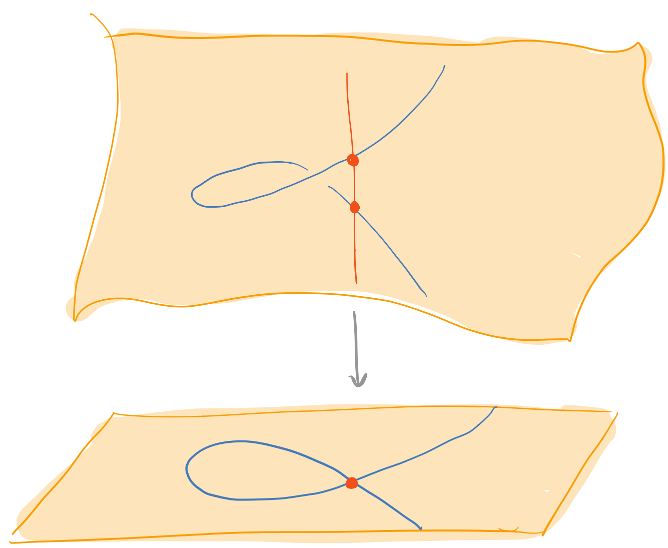
# Blowing up a point in $\mathbb{A}^2$

"Resolving the singularities" of a curve $C$ means finding a smooth projective curve $X$ and a birational morphism

$$f : X \longrightarrow C.$$

In particular, the points on $X$ correspond to DVRs of $k(C)$.



Rough idea of the strategy: let $C \subseteq \mathbb{P}^2$ be a curve, $P \in C$ a singular point. Remove $P$ from $\mathbb{P}^2$ and replace it w/ a $\mathbb{P}^1$, w/ each point corresponding to a tangent direction at $P$. The new curve will have better singularities (i.e. pts will have lower multiplicities).

First, we'll show how to blow up a point in $\mathbb{A}^2$, replacing it with an $\mathbb{A}^1$. For this, we need the following definition:

**Def:** If $f : X \longrightarrow Y$ is a morphism of varieties, the graph of $f$ is

$$G(f) := \{ (x, y) \in X \times Y \mid y = f(x) \}$$

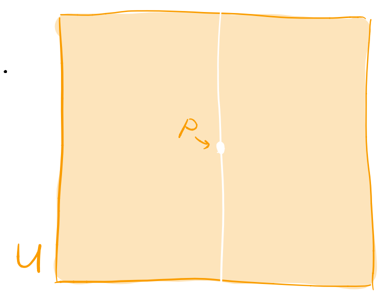
This is in fact a closed subvariety of $X \times Y$ and the projection $G(f) \to X$ is an isomorphism.

**Note:** The Zariski topology on $X \times Y$ is inherited by the $(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \times \mathbb{A}^m) \times (\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_s} \times \mathbb{A}^p)$ it sits inside. This is not the product topology!

Since we can always move a singular point to the origin, we only need to know how to blow up $\mathbb{A}^2$ at the origin:

**Directions for blowing up $\mathbb{A}^2$ at $(0,0)$:**

1. Let $P = (0,0)$, $U = \{(x,y) \in \mathbb{A}^2 \mid x \neq 0\}$. Define the morphism $f : U \to \mathbb{A}^1 = k$ by $f(x,y) = y/x$.




2. Let $G \subseteq U \times \mathbb{A}^1 \subseteq \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$ be the graph of $f$. So $G = \{(x,y,z) \in \mathbb{A}^3 \mid y = xz, x \neq 0\}$

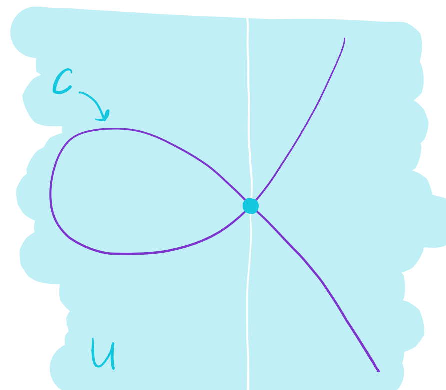
3. Let $B = \{(x,y,z) \in \mathbb{A}^3 \mid y = xz\}$, i.e. the closure of $G$ in $\mathbb{A}^3$. $xz - y$ is irreducible, so $B$ is an (affine) variety.

4. Let $\pi : B \to \mathbb{A}^2$ be the projection $\pi(x,y,z) = (x,y)$. Then $\pi(B) = U \cup \{P\}$.

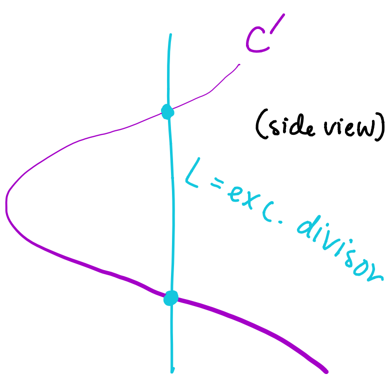
⑤ Let $L = \pi^{-1}(P) = \{(0,0,z) \mid z \in k\}$. $\pi^{-1}(U) = G$, so $\pi$ restricts to an isomorphism of $\pi^{-1}(U)$ onto $U$.

$\pi$ is locally the blowup of $\mathbb{A}^2$. Soon we will look at the projective case, where the equations are more complicated, but it will locally look like this.

**Ex:** Let $C = V(y^2 - x^2(x+1))$.




$\pi^{-1}(C)$ will be the union of a line, $L$ (called the exceptional divisor) and then a curve $C'$ that is birational to $C$ (called the strict transform of $C$). To get $C'$, we take the closure of the preimage of $C - \{P\}$, i.e. $\overline{\pi^{-1}(C - \{P\})}$.




Since $\pi^{-1}(C - \{P\}) \subseteq G$, it is isomorphic to $C'$.

To figure out what $C'$ is and which points lie over $P$, we project onto the $xz$-plane.

Let $\varphi : \mathbb{A}^2 \longrightarrow B$ be the isomorphism $\varphi(x,z) = (x, xz, z)$.

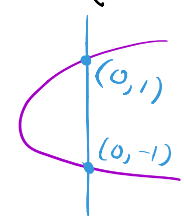
(Projection onto $xz$-plane gives the inverse.) Then the map

$$\psi = \pi \circ \varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2 \quad \text{is} \quad \psi(x,z) = (x, xz)$$

and is birational!

Then the preimage of $C$ is $\psi^{-1}(C) = V\big((xz)^2 - x^2(x+1)\big)$

$$x^2z^2 - x^2(x+1) = \underset{\uparrow \atop \text{Exc. divisor}}{x^2}\underbrace{(z^2 - x - 1)}_{C'}.$$




So $C$ is birational to the parabola $V(z^2 - x - 1)$, and the points lying over $P$ are $\{(x,z) \mid (x, xz) = (0,0)\}$ i.e. $(0,1)$ and $(0,-1)$

More generally: Let $C \subseteq \mathbb{A}^2$ be an irreducible curve not equal to the $y$-axis. Let $C_0 = C \cap U$, an open subvariety of $C$.

Let $C_0' = \psi^{-1}(C_0)$, and $C'$ be the closure of $C_0'$ in $\mathbb{A}^2$. Let $f : C' \longrightarrow C$ be the restriction of $\psi$ to $C'$. Then $f$ is a birational morphism, just as above.

Let $C = V(g)$, $g$ irreducible, and $g = g_r + g_{r+1} + \dots + g_n$, $g_i$ a form of deg $i$, $r = m_P(C)$.

Claim: $C' = V(g')$, where $g' = g_r(1,z) + x\, g_{r+1}(1,z) + \dots + x^{n-r} g_n(1,z)$.

**Pf:** $g(x, xz) = x^r g_r(1,z) + x^{r+1} g_{r+1}(1,z) + \cdots = x^r g'$.
Since $g_r(1,z) \neq 0$, $x$ doesn't divide $g'$.

If $g' = ab$ then $g = x^r a(x, y/x)\, b(x, y/x)$ would be reducible.
Thus, $g'$ is irreducible and $V(g') \supseteq C_0'$, $V(g') = C'$. $\square$

Now, using the same setup as above, we want to show that the preimage $\varphi^{-1}(P)$ consists of a unique point for each tangent direction of $C$ at $P$.

Since we are working only in $U$, which doesn't contain the $y$-axis, we can assume $V(x)$ is not tangent to $C$ at $P$ (possibly after a change of coordinates). We won't need to make this assumption in the projective case.

Then, after multiplying $g$ by a constant, we can write

$$g_r = \prod_{i=1}^{s} (y - \alpha_i x)^{r_i}$$

where $y - a_i x$ are the tangents to $C$ at $P$.

**Claim:** $f^{-1}(P) = \{P_1, \ldots, P_s\}$, where $P_i = (0, \alpha_i)$ and

$$m_{P_i}(C') \leq I_{P_i}(C', \underset{\substack{\uparrow \\ \text{exc. divisor} \\ = V(x)}}{E}) = r_i.$$

Thus, if $P$ is an ordinary multiple pt on $C$ (i.e. each $r_i = 1$) then each $P_i$ is a simple point on $C'$, and $\mathrm{ord}_{P_i}^{C'}(x) = 1$.

**Pf:** $f^{-1}(P) = C' \cap E = \{(0,\alpha) \mid \underbrace{g_r(1,\alpha) = 0}_{\text{since } x \text{ divides every other summand}}\}$.

And $m_{P_i}(C') \le I_{P_i}(g', x)$

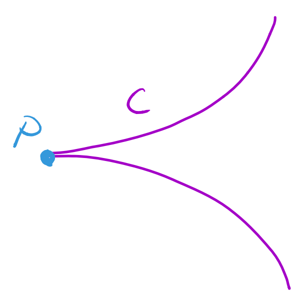
$$= I_{P_i}(f_r(1,z), x)$$

$$= I_{P_i}\Big(\prod_{j=1}^{s}(z-\alpha_j)^{r}, x\Big) = I_{P_i}\big((z-\alpha_i)^{r_i}, x\big) = r_i. \quad \square$$

other factors don't pass through $P_i$

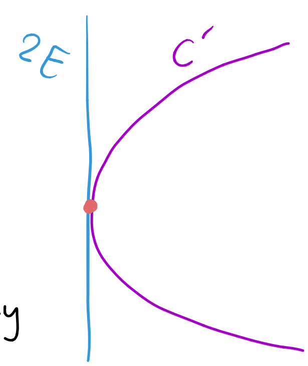
**Q:** What happens to a singular point w/ only one tangent line, i.e. a tangent line w/ high multiplicity?

**Ex:** 1.) Let $C = V(y^2 - x^3)$




Using the same strategy as above, we blow up the origin using the map $\Psi : \mathbb{A}^2 \to \mathbb{A}^2$ defined

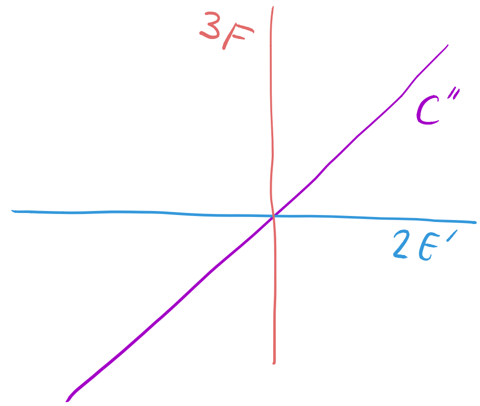
$(x,z) \mapsto (x, xz)$




The preimage of $C$ is then cut out by

$(xz)^2 - x^3 = x^2(z^2 - x)$.

When resolving curve singularities, we usually want the strict transform to intersect the exceptional divisor transversally, so we blow up again. However, our tangent line is now $x=0$, so we do an affine change of coords first to get

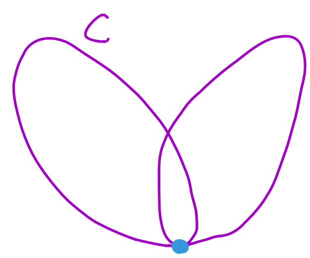



$z^2(x^2-z)$. The blowup gives

$$(xw)^2(x^2-xw) = \underbrace{x^3}_{\substack{\text{new}\\ \text{exc.}\\ \text{div.}}}\ \underbrace{w^2}_{\substack{\text{strict}\\ \text{trans.}\\ \text{of } E}}\ \underbrace{(x-w)}_{\substack{\text{strict}\\ \text{transform}\\ \text{of } C'}}$$

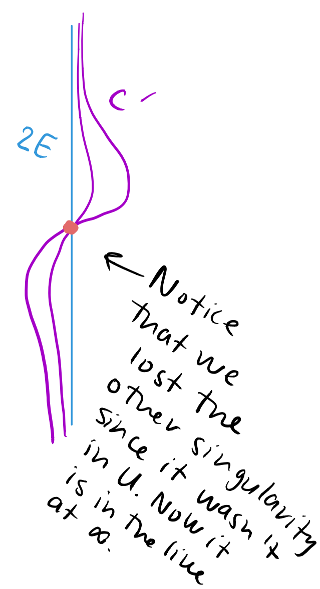
So $C$ is birational to a $\mathbb{P}^1$, which we already knew

2.) Now let $C = V(2x^4 - 3x^2y + y^2 - 2y^3 + y^4)$. Let's resolve the singularity at the origin. (This kind of singularity is called a tacnode)




The preimage under the blowup becomes

$$2x^4 - 3x^3z + x^2z^2 - 2x^3z^3 + x^4z^4$$
$$= x^2\Big(\underbrace{2x^2 - 3xz + z^2}_{\substack{\text{tangent lines}\\ (x-z)(2x-z)}} - 2xz^3 + x^2z^4\Big)$$

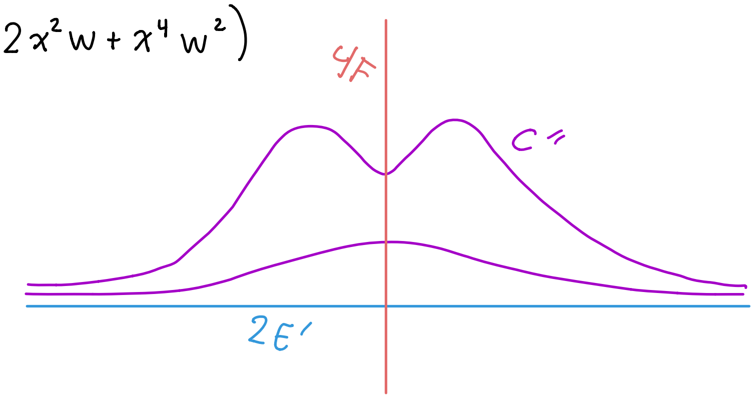



A change of coordinates gives

$$z^2(2z^2 - 3xz + x^2 - 2x^3z + x^4z^2).$$

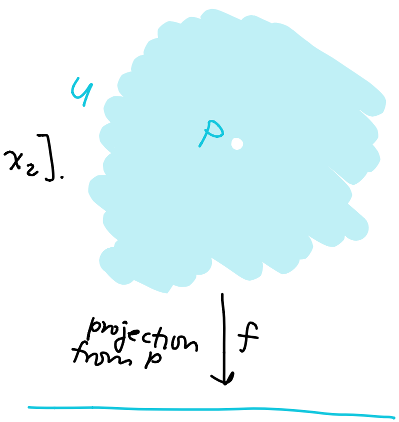
Blowing up again yields

$$x^2w^2\left(2x^2w^2 - 3x^2w + x^2 - 2x^4w + x^6w^2\right)$$
$$= x^4w^2\left(2w^2 - 3w + 1 - 2x^2w + x^4w^2\right)$$




## Blowing up $\mathbb{P}^2$ at a point

Rough idea: let $P = [0:0:1]$, $U = \mathbb{P}^2 \setminus P$.




Define $f : U \longrightarrow \mathbb{P}^1$ by $f[x_1 : x_2 : x_3] = [x_1 : x_2]$.

Let $G \subseteq U \times \mathbb{P}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ be the graph of $f$.

Let $y_1, y_2$ be the homog. coords for $\mathbb{P}^1$, $x_1, x_2, x_3$ the homog. coords for $\mathbb{P}^2$. Then define:

$$B = V(y_1x_2 - y_2x_1) \subseteq \mathbb{P}^2 \times \mathbb{P}^1$$

Then $G \subseteq B$. In fact, $B \setminus G = \{([0:0:1], [x_1 : x_2]) \mid [x_1 : x_2] \in \mathbb{P}^1\} = E$

In particular, $B = \overline{G}$. exc. divisor

Let $\pi : B \longrightarrow \mathbb{P}^2$ be the projection onto the first factor.

This is called the <u>blowup</u> of $\mathbb{P}^2$ at $[0:0:1]$, and is isom. when restricted to $B \setminus E \to U$.

Locally, this is the same map as in the affine case.

Suppose we want to study an affine neighborhood of a point $Q = [1:\lambda] \in E$:

Let $\varphi_3 : \mathbb{A}^2 \to U_3 \subseteq \mathbb{P}^2$ be the morphism $(x,y) \longmapsto [x:y:1]$.

Let $\psi : \mathbb{A}^2 \to \mathbb{A}^2$ be as above: $(x,z) \mapsto (x, xz)$.

Define $\varphi : \mathbb{A}^2 \to \mathbb{P}^2 \times \mathbb{P}^1$ by

$$\varphi(x,z) = [x : xz : 1] \times [1 : z] \in B$$

$$\begin{tikzcd}
(x,z) \arrow[rrr, mapsto] \arrow[ddd, mapsto, bend right] & & & {[x:xz:1]\times[1:z]} \arrow[ddd, mapsto] \\
& \mathbb{A}^2 \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \arrow[d, "\pi"] & \\
& \mathbb{A}^2 \arrow[r, hook, "\varphi^3"'] & \mathbb{P}^2 & \\
(x,xz) \arrow[rrr, mapsto, bend right] & & & {[x:xz:1]}
\end{tikzcd}$$

Then $\varphi(\mathbb{A}^2)$ is a neighborhood of $Q$, so we can just work with $\psi : \mathbb{A}^2 \to \mathbb{A}^2$ instead of $\pi$.