Leskelä–Alaluusua

Examination

You may bring to the exam a handwritten *memory aid sheet* of size A4 with text only on one side, containing your name and student id in the upper right corner. You don't need to return your memory aid sheet. The exam contains 4 problems each worth 6 points.

1. Which of the following statements are true in general for real-valued random variables? Explain why a statement is generally true, or give a counterexample.

(a) $X \perp\!\!\!\perp Y$ and $Y \perp\!\!\!\perp Z \implies X \perp\!\!\!\perp Z$. **(2 p)**
Solution. False. Let X and Y be independent and Ber(1/2)-distributed, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Define $Z(\omega) = X(\omega)$ for all ω. Then $X \perp\!\!\!\perp Y$ and $Y \perp\!\!\!\perp Z$ but X and Z are not independent.

For example, we may take $\Omega = \{0,1\}^2$ and $\mathcal{A} = 2^\Omega$ and $\mathbb{P}$ to be the uniform distribution on Ω, and then define $X(\omega_1,\omega_2) = \omega_1$, $Y(\omega_1,\omega_2) = \omega_2$, and $Z(\omega_1,\omega_2) = \omega_1$.

(b) $(X_1, X_2) \perp\!\!\!\perp (Y_1, Y_2) \implies X_1 \perp\!\!\!\perp Y_2$. **(2 p)**
Solution. True. Define $f(x_1,x_2) = x_1$ and $g(y_1,y_2) = y_2$. Define $X = (X_1,X_2)$ and $Y = (Y_1,Y_2)$. Then $X \perp\!\!\!\perp Y$ implies $f(X) \perp\!\!\!\perp g(Y)$, as we remember from the lectures.

(c) $X \perp\!\!\!\perp Y$ and $X, Y \geq 0 \implies \mathbb{E}(XY) = \mathbb{E}X\,\mathbb{E}Y$. **(2 p)**
Solution. True. This was proved in the lectures. For indicators $X = 1_A$ and $Y = 1_B$ this reduces to the definition of independence. By linearity this extends to finite-range random variables. By monotone continuity of integration, this in turn extends to nonnegative random variables, be they finite or infinite.

2. Fix numbers $0 < a < b$, and let $S_n = X_1 + \cdots + X_n$ be a sum of independent random variables distributed according to the exponential distribution with rate parameter b and density function $f_b(x) = 1_{(0,\infty)}(x)\, be^{-bx}$ with respect to the Lebesgue measure on the real line.

(a) Compute the moment generating function $M_{S_n}(t) = \mathbb{E}e^{tS_n}$. For which values of t is the moment generating function finite? **(2 p)**
Solution.

$$M_{X_1}(t) \;=\; \int_0^\infty e^{tx}\, be^{-bx}\, dx \;=\; \int_0^\infty be^{(t-b)x}\, dx \;=\; \begin{cases} \frac{b}{b-t}, & t < b, \\ \infty, & t \geq b. \end{cases}$$

By independence,

$$M_{S_n}(t) \;=\; \mathbb{E}\prod_{j=1}^n e^{tX_j} \;=\; M_{X_1}(t)^n \;=\; \begin{cases} \left(\frac{b}{b-t}\right)^n, & t < b, \\ \infty, & t \geq b. \end{cases}$$

(b) Prove that $\mathbb{P}(\frac{1}{n}S_n \geq \frac{1}{a}) \leq e^{-nt/a} M_{S_n}(t)$ for all $t > 0$. **(1 p)**
Solution. By Markov's inequality,

$$\mathbb{P}\Big(\frac{1}{n}S_n \geq \frac{1}{a}\Big) \;=\; \mathbb{P}\Big(S_n \geq \frac{n}{a}\Big) \;=\; \mathbb{P}\Big(e^{tS_n} \geq e^{nt/a}\Big) \;\leq\; \frac{1}{e^{nt/a}}\mathbb{E}e^{tS_n} \;=\; e^{-nt/a}M_{S_n}(t).$$

(c) Determine a value of t that yields the sharpest bound in (b) and prove that

$$\mathbb{P}\left(\frac{1}{n}S_n \geq \frac{1}{a}\right) \;\leq\; e^{-nD(a\|b)},$$

where $D(a\|b) = \log\frac{a}{b} + \frac{b}{a} - 1$. **(2 p)**
Solution. For any $0 < t < b$,

$$e^{-nt/a}M_{S_n}(t) \;=\; e^{-nt/a}\left(\frac{b}{b-t}\right)^n \;=\; e^{-nI(t)},$$

where $I(t) = t/a - \log(\frac{b}{b-t})$. Differentiation shows that

$$I'(t) \;=\; \frac{1}{a} - \frac{1}{b-t}, \qquad I''(t) \;=\; -\frac{1}{(b-t)^2}.$$

We see that I is strictly concave on $(0, b)$ and $I'(t_*) = 0$ for $t_* = b - a$. This is the point at which I attains its largest value, and we obtain the sharpest bound in (c). By substituting this value in (c) we see that $\mathbb{P}\Big(\frac{1}{n}S_n \geq \frac{1}{a}\Big) \leq e^{-nI(t_*)}$ with

$$I(t_*) \;=\; t_*/a - \log\frac{b}{b-t_*} \;=\; (b-a)/a - \log\frac{b}{a} \;=\; \log\frac{a}{b} + \frac{b}{a} - 1.$$

(d) Comment briefly how the inequality in (c) relates to the weak law of large numbers. **(1 p)**
Solution. We note that $\mathbb{E}X_1 = \frac{1}{b}$. The weak law of large numbers hence states that $\frac{1}{n}S_n \to \frac{1}{b}$ in probability. As a consequence, for $\epsilon = \frac{1}{a} - \frac{1}{b} > 0$,

$$\mathbb{P}\Big(\frac{1}{n}S_n \geq \frac{1}{a}\Big) \;=\; \mathbb{P}\Big(\frac{1}{n}S_n - \frac{1}{b} \geq \epsilon\Big) \;\leq\; \mathbb{P}\Big(\Big|\frac{1}{n}S_n - \frac{1}{b}\Big| \geq \epsilon\Big) \;\to\; 0.$$

Therefore, the WLLN tells that $\mathbb{P}(\frac{1}{n}S_n \geq \frac{1}{a}) \to 0$, whereas the upper bound of (c) tells that this convergence happens exponentially fast.

3. Let X_n, X be a real-valued random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{E}|X| < \infty$ and $\mathbb{E}|X_n - X| \le \frac{1}{n}$ for all $n \ge 1$. For each of the following, prove that the statement is true, or give a counterexample confirming that the statement is false.

(a) $X_n \xrightarrow{\mathbb{P}} X$. **(2 p)**
Solution. True. Markov's inequality implies $\mathbb{P}(|X_n - X| > \epsilon) \le \epsilon^{-1}\mathbb{E}|X_n - X| \le \epsilon^{-1} n^{-1}$ for all $\epsilon > 0$.

(b) $\mathbb{P}(X_n = 0) \to \mathbb{P}(X = 0)$. **(2 p)**
Solution. False. Consider constant random variables such that $X_n(\omega) = 1/n$ and $X(\omega) = 0$ for all ω. Then $\mathbb{E}|X_n - X| = \frac{1}{n} \to 0$ but

$$\big|\mathbb{P}(X_n = 0) - \mathbb{P}(X = 0)\big| \;=\; |1 - 0| \;=\; 1 \qquad \text{for all } n.$$

(c) $W_1(\mathrm{Law}(X_n), \mathrm{Law}(X)) \to 0$. **(2 p)**
Solution. True. Denote $\mu_n = \mathrm{Law}(X_n)$, $\mu = \mathrm{Law}(X)$, and $\lambda_n = \mathrm{Law}(X_n, X)$. Then the marginals of λ_n are equal to μ_n and μ, so that $\lambda_n \in \Gamma(\mu_n, \mu)$. Therefore,

$$W_1(\mu_n, \mu) \;\le\; \int_{\mathbb{R}^2} |x - y| \, \lambda_n(dx, dy) \;=\; \mathbb{E}|X_n - X| \;\le\; \frac{1}{n} \;\to\; 0.$$

Recall that the Wasserstein distance between probability measures μ and ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is defined by

$$W_1(\mu, \nu) \;=\; \inf_{\lambda \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^2} |x - y| \, \lambda(dx, dy),$$

with $\Gamma(\mu, \nu)$ denoting the collection of probability measures on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ having first marginal μ and second marginal ν.

4. For a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, define $(\mu K)(B) = \int_{\mathbb{R}} K(x, B) \, \mu(dx)$, where

$$K(x, B) \;=\; \sum_{y=0}^{\infty} 1_B(x + y) \, e^{-5} \frac{5^y}{y!} \qquad \text{for } x \in \mathbb{R} \text{ and } B \in \mathcal{B}(\mathbb{R}).$$

(a) Prove that μK is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. **(2 p)**
Solution. Denote $k(y) = e^{-5} \frac{5^y}{y!}$. For disjoint $B_1, B_2, \dots \in \mathcal{B}(\mathbb{R})$, $1_{\cup_j B_j}(x + y) = \sum_j 1_{B_j}(x + y)$ implies (F refers to Fubini)

$$K(x, \cup_j B_j) \;=\; \sum_{y=0}^{\infty} \sum_{j=1}^{\infty} 1_{B_j}(x + y) \, k(y) \;\overset{\mathrm{F}}{=}\; \sum_{j=1}^{\infty} \sum_{y=0}^{\infty} 1_{B_j}(x + y) \, k(y) \;=\; \sum_{j=1}^{\infty} K(x, B_j).$$

Hence

$$\int_{\mathbb{R}} K(x, \cup_j B_j)\, \mu(dx) \;=\; \int_{\mathbb{R}} \sum_{j=1}^{\infty} K(x, B_j)\, \mu(dx)$$
$$\overset{\text{F}}{=} \;\sum_{j=1}^{\infty} \int_{\mathbb{R}} K(x, B_j)\, \mu(dx) \;=\; \sum_{j=1}^{\infty} (\mu K)(B_j),$$

so that μK is disjointly sigma-additive.

Hence μK is a measure because $(\mu K)(\emptyset) = \int_{\mathbb{R}} K(x, \emptyset)\, \mu(dx) = \int_{\mathbb{R}} 0\, \mu(dx) = 0$, and a probability measure because $(\mu K)(\mathbb{R}) = \int_{\mathbb{R}} K(x, \mathbb{R})\, \mu(dx) = \int_{\mathbb{R}} 1\, \mu(dx) = 1$.

(b) Compute $\mu_2(\{0\})$ where $\mu_2 = (\delta_0 K)K$ and δ_0 is the Dirac measure at 0. **(2 p)**
Solution. Let $\mu_1 = \delta_0 K$. Then

$$\mu_1(B) \;=\; (\delta_0 K)(B) \;=\; \int_{\mathbb{R}} K(x, B)\, \delta_0(dx) \;=\; K(0, B),$$

so that

$$\mu_1(B) \;=\; \sum_{y=0}^{\infty} 1_B(y)\, k(y) \;=\; \sum_{y=0}^{\infty} k(y)\, \delta_y(B)$$

equals the Poisson distribution with probability mass function $k(y) = e^{-5}\frac{5^y}{y!}$. Because μ_1 is a discrete probability measure with all its mass concentrated on the set of nonnegative integers, we find that

$$\mu_2(B) \;=\; \int_{\mathbb{R}} K(y, B)\, \mu_1(dy) \;=\; \sum_{y=0}^{\infty} K(y, B)\, k(y).$$

In particular,

$$\mu_2(\{0\}) \;=\; \sum_{y=0}^{\infty} K(y, \{0\}) k(y) \;=\; \sum_{y=0}^{\infty} \left(\sum_{z=0}^{\infty} 1_{\{0\}}(y + z)\, k(z) \right) k(y) \;=\; k(0)^2 \;=\; e^{-10}.$$

Solution. [Alternative solution] The kernel represents a random walk where at each step we add a Poisson distributed random variable with mean 5. The initial condition $\mu_0 = \delta_0$ tells that the random walk starts deterministically at 0. After two steps the walk has made two independent Poisson jumps. Hence $\mu_2 = \text{Law}(0 + \xi_1 + \xi_2)$ where ξ_1, ξ_2 are independent Poisson-distributed random variables with mean 5. Then $\mu_2(\{0\}) = \mathbb{P}(0 + \xi_1 + \xi_2 = 0) = \mathbb{P}(\xi_1 = 0, \, \xi_2 = 0) = \mathbb{P}(\xi_1 = 0)\, \mathbb{P}(\xi_2 = 0) = e^{-10}$.

(c) Let $\gamma_1 = \gamma_0 K$ where $\gamma_0(B) = \int_B \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$ is the standard normal distribution. Determine independent random variables X and Y such that $\gamma_1 = \text{Law}(X + Y)$. **(2 p)**

Solution. Let X and Y be independent and such that X is standard normal and Y is Poisson-distributed with mean 5. Then $\mathrm{Law}(X) = \gamma_0$ and $\mathrm{Law}(Y) = \delta_0 K =: \mu_1$. Then

$$\begin{aligned}
\mathbb{P}(X+Y \in B) &= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_B(x+y)\, \gamma_0(dx)\, \mu_1(dy) \\
&= \int_{\mathbb{R}} \left(\sum_{y \in \mathbb{Z}_+} 1_B(x+y)\, \mu_1(\{y\}) \right) \gamma_0(dx) \\
&= \int_{\mathbb{R}} K(x,B)\, \gamma_0(dx) \\
&= (\gamma_0 K)(B).
\end{aligned}$$

Therefore, we conclude that $\mathrm{Law}(X+Y) = \gamma_0 K$.