Chapter 5

Exercises covered and hints:

2.

i) Prove that $\mathfrak{p}, \mathfrak{q}$ are maximal ideals, hence prime.

To prove $\mathfrak{p}, \mathfrak{q}$ are maximal consider the following method (which might be simpler than the method used in the examples):

- $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/\langle x^2+5\rangle = R$ by considering

$$\varphi : \mathbb{Z}[\sqrt{-5}] \to R$$
$$\sqrt{-5} \mapsto x + \langle x^2+5\rangle$$

- $\mathfrak{p}$ is maximal in $\mathbb{Z}[\sqrt{-5}]$ iff $\varphi(\mathfrak{p})$ is maximal in R iff $R/\varphi(\mathfrak{p})$ is a field
- Prove $R/\varphi(\mathfrak{p})$ is a field

ii) Show that $\mathfrak{p}^2 = \langle 2\rangle$ and $\mathfrak{p}\mathfrak{q} = \langle 1+\sqrt{-5}\rangle$.

iii) Show that the factorizations of 6, $6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$, come from two different groupings of the factorization into prime ideals $\langle 6\rangle = \mathfrak{p}^2\mathfrak{q}\mathfrak{r}$.

3.

i) $N(\mathfrak{p})$?
ii) $N(\mathfrak{q})$?

$\Big]$ Recall $N(\mathfrak{p}) := |\mathcal{O}/\mathfrak{p}|$
Again use $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/\langle x^2+5\rangle$

iii) $N(\mathfrak{p}^2)$?
iv) $N(\mathfrak{p}\mathfrak{q})$?

$\Big]$ Corollary 5.10

12. See example 5.18.

6. Consider $\mathcal{J}_2 = \langle \sqrt{-3} \rangle$.